

CALOGERO-MOSER SYSTEMS IN $SU(N)$ SEIBERG-WITTEN THEORY ***Eric D'Hoker ¹ and D.H. Phong ²**¹ Department of Physics
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Columbia University, New York, NY 10027, USA**ABSTRACT**

The Seiberg-Witten curve and differential for $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory, with a massive hypermultiplet in the adjoint representation of the gauge group, are analyzed in terms of the elliptic Calogero-Moser integrable system. A new parametrization for the Calogero-Moser spectral curves is found, which exhibits the classical vacuum expectation values of the scalar field of the gauge multiplet. The one-loop perturbative correction to the effective prepotential is evaluated explicitly, and found to agree with quantum field theory predictions. A renormalization group equation for the variation with respect to the coupling is derived for the effective prepotential, and may be evaluated in a weak coupling series using residue methods only. This gives a simple and efficient algorithm for the instanton corrections to the effective prepotential to any order. The 1- and 2- instanton corrections are derived explicitly. Finally, it is shown that certain decoupling limits yield $\mathcal{N} = 2$ supersymmetric theories for simple gauge groups $SU(N_1)$ with hypermultiplets in the fundamental representation, while others yield theories for product gauge groups $SU(N_1) \times \cdots \times SU(N_p)$, with hypermultiplets in fundamental and bi-fundamental representations. The spectral curves obtained this way for these models agree with the ones proposed by Witten using D-branes and M-theory.

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I. INTRODUCTION

The low energy effective theory for $\mathcal{N} = 2$ supersymmetric gauge theories, with a hypermultiplet in the adjoint representation of the gauge group, was constructed for $SU(2)$ gauge group in the original paper by Seiberg and Witten [1]. It was subsequently constructed for arbitrary $SU(N)$ gauge groups by Donagi and Witten [2], who also showed on general grounds that $\mathcal{N} = 2$ supersymmetric gauge theories are described by integrable Hamiltonian systems. In particular, the integrable system corresponding to the $SU(N)$ theory with matter in the adjoint representation was identified as an $SU(N)$ Hitchin system, that is, a completely integrable model arising from a two-dimensional $SU(N)$ gauge theory. A possible close relation between the spectral curves proposed by Donagi and Witten and the spectral curves of elliptic Calogero-Moser systems [3] was suggested by Martinec in [4], and established explicitly by Krichever in unpublished work.

The goal of the present paper is to analyze carefully the $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory with a hypermultiplet in the adjoint representation, using the elliptic Calogero-Moser integrable system. This is the dynamical system

$$p_i = \dot{x}_i, \quad \dot{p}_i = m^2 \sum_{j \neq i} \wp'(x_i - x_j), \quad 1 \leq i, j \leq N. \quad (1.1)$$

where $\wp(z)$ is the Weierstrass \wp -function. It admits the Lax representation $\dot{L} = [M, L]$, with $L(z)$, $M(z)$ given by $N \times N$ matrices depending both on the dynamical variables x_i, p_i and a spectral parameter z living on a torus Σ (c.f. (2.11) below). The complex modulus τ of the torus Σ is related to the gauge coupling e and the θ -angle of the gauge theory by

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}. \quad (1.2)$$

The spectral curve of the elliptic Calogero-Moser system is given by

$$R(k, z) \equiv \det(kI - L(z)) = 0 \quad (1.3)$$

and provides the curve of the $\mathcal{N} = 2$ theory, while the Seiberg-Witten differential is naturally constructed out of the spectral parameter z by $d\lambda = kdz$. This construction fits naturally in the general framework for integrable systems and Seiberg-Witten differentials proposed in [5][6] (c.f. Section II).

The case of the adjoint representation presents however a number of distinctive features which set it apart from the case of the fundamental representation, for which a

well-developed machinery is now available [7,8,9] (see also [10], where an extensive list of references can be found). Indeed, the spectral curves in the fundamental representation case can be motivated by a weak-coupling analysis, in which the classical order parameters of the four-dimensional gauge theory (i.e. the vacuum expectation values of the scalar field belonging to the gauge multiplet) are apparent. In the adjoint representation case, there is at the present time no such argument. Rather, the match between the four-dimensional gauge theory and the two-dimensional integrable model was found via indirect arguments, where the order parameters are difficult to recognize even in the weak-coupling limit.

From the viewpoint of Calogero-Moser systems, the N -dimensional family of spectral curves (1.3) is parametrized by an overdetermined set of $2N$ phase variables x_i, p_i , and it is difficult to identify in terms of the x_i, p_i the crucial N variables which correspond to the order parameters of the gauge theory. Thus a key ingredient of our approach is a convenient parametrization of the Calogero-Moser spectral curves by a single monic polynomial $H(k) = \prod_{i=1}^N (k - k_i)$ of degree N , whose zeroes k_i are essentially the classical order parameters of the gauge theory. More precisely, if we set

$$f(k, z) \equiv R(k - m\partial_z \log \vartheta_1(\frac{z}{2\omega_1}|\tau), z),$$

then the elliptic Calogero-Moser spectral curves are characterized by

$$f(k, z) = \frac{1}{\vartheta_1(\frac{z}{2\omega_1}|\tau)} \vartheta_1(\frac{1}{2\omega_1} \{z - m\frac{\partial}{\partial k}\}|\tau) H(k), \quad (1.4)$$

and the classical order parameters are given by

$$\lim_{q \rightarrow 0} \frac{1}{2\pi i} \oint_{A_i} d\lambda = k_i - \frac{1}{2}m, \quad (1.5)$$

where m is the mass of the hypermultiplet. This is established in Section III.

With the parametrization (1.4), it is then possible to set up a systematic expansion at weak gauge coupling of the quantum order parameters and their duals, and thus of the effective prepotential. Using the methods developed in [7], we derive in this way the perturbative contribution $\mathcal{F}^{\text{pert}}$ to the effective prepotential

$$\mathcal{F}^{\text{pert}} = -\frac{1}{8\pi i} \sum_{i,j=1}^N \{(a_i - a_j)^2 \log(a_i - a_j)^2 - (a_i - a_j + m)^2 \log(a_i - a_j + m)^2\} \quad (1.6)$$

It agrees with the well-known formula obtained from direct quantum field theory calculations, and confirms that the integrable model corresponding to the $\mathcal{N} = 2$ supersymmetric

gauge theory with matter in the adjoint representation is indeed the elliptic Calogero-Moser system. We also evaluate the 1-instanton correction, whose calculation has not yet been performed by direct instanton methods (Section IV).

In Section V, we derive a renormalization group (RG) type equation for the variation of the prepotential with respect to the complex gauge coupling τ :

$$\frac{\partial \mathcal{F}}{\partial \tau} = \frac{1}{4\pi i} \sum_{j=1}^N \oint_{A_j} k^2 dz \quad (1.7)$$

Remarkably, the right hand side coincides with the Hamiltonian of the Calogero-Moser system. Such a relation also exists for theories with matter in the fundamental representation [11][6]. It suggests a more direct link between $\mathcal{N} = 2$ supersymmetric gauge theories and integrable models than just coincidence of spectral curves: *the RG beta function of the four-dimensional theory is given by the Hamiltonian of the two-dimensional integrable model*. An important feature of (1.7) is that the beta function may be evaluated in a series expansion at weak coupling, using residue methods only. As a consequence, instanton corrections can be routinely calculated to an arbitrarily high order. We illustrate the process by deriving both the 1-instanton and the 2-instanton corrections to the prepotential. As expected, the 1-instanton answer agrees with the formula obtained earlier in Section IV.

Finally, in Section VI, we take various decoupling limits in which the hypermultiplet mass parameter m is taken to infinity. The simplest limit is when the full hypermultiplet is decoupled, yielding the pure $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory. We verify that the curve as well as the effective prepotential converge towards the forms for the pure theory. However, a number of more subtle limits may be taken, in which some of the order parameters are taken to infinity as well. In general, this will yield an $\mathcal{N} = 2$ supersymmetric theory whose gauge group is a product $SU(N_1) \times \cdots \times SU(N_p)$ with hypermultiplets in fundamental and bi-fundamental representations of the gauge group. These theories appear to be special cases of the general models with product $SU(N)$ groups which were recently solved by Witten using M-theory and D-brane technology [12] (see also the recent work of Katz, Mayr, and Vafa [25] based rather on compactifications of Type II strings on Calabi-Yau manifolds). Our models appear with specific relations between the N_i , as well as between their gauge couplings. In particular, we recover in this way theories with gauge group $SU(N_c)$ and with N_f hypermultiplets in the fundamental representation. We note that, conversely, Witten has shown in [12] how to recover the curves of [2] from M-theory.

II. THE SPECTRAL CURVES FOR THE ADJOINT REPRESENTATION

The main goal of this section is to provide a survey of the several equivalent descriptions of the candidates for the spectral curves of the $\mathcal{N} = 2$ supersymmetric gauge theory with a hypermultiplet in the adjoint representation.

We begin by reviewing the original description and motivation of Donagi and Witten [2]. The idea in their work is to view the theory as an $\mathcal{N} = 4$ theory with a bare mass term breaking the $\mathcal{N} = 4$ supersymmetry to an $\mathcal{N} = 2$ supersymmetry, and to deform correspondingly the spectral curves for the $\mathcal{N} = 4$ theory. In the Coulomb phase of $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theories, the moduli space of vacua is parametrized by the vacuum configurations of the scalar field ϕ in the $\mathcal{N} = 2$ gauge multiplet. At the vacuum, ϕ is constant and lies in the Cartan subalgebra, $\phi = \sum_{i=1}^{N-1} a_i H_i$. In the $\mathcal{N} = 4$ supersymmetric case, the metric on the space of vacua $\{a_i\}$ receives no quantum corrections and is given by

$$ds^2 = (\text{Im } \tau) \sum_{i=1}^N da_i \overline{da_i} \quad (2.1)$$

where τ is the microscopic gauge coupling. A key observation of [2] is that this metric also arises by a Hitchin construction, i.e., an integrable model based on a *two-dimensional* gauge theory [13]. Let Σ be the torus with modulus τ , and define the space X by

$$X = \{(A, \Phi); \overline{\nabla}_A \varphi = 0\} / H, \quad (2.2)$$

where A is a connection on a flat $SU(N)$ bundle over Σ , φ is a $(1,0)$ form, both with values in the adjoint representation, and H is the group of (complexified) gauge transformations. Since the space of connections modulo H is the space \mathcal{M} of holomorphic $SL(n, \mathbf{C})$ bundles over Σ , and since the cotangent space of \mathcal{M} is the space of holomorphic one-forms valued in the adjoint representation, X can be recognized as $T^*(\mathcal{M})$. As such, it admits a symplectic form ω . More concretely, the fundamental group of Σ is abelian, and \mathcal{M} is the same as the moduli space of holomorphic $U(1)^{N-1}$ bundles over Σ , divided by the Weyl group W , $\mathcal{M} = \Sigma^{N-1} / W$. At generic points of \mathcal{M} , it follows from the condition $\overline{\nabla}_A \varphi = 0$ that φ reduces to a constant function with values in the Cartan subalgebra \mathbf{C}^{N-1} . Thus we can write

$$X = (\Sigma^{N-1} \times \mathbf{C}^{N-1}) / W \quad (2.3)$$

If we represent each copy of Σ in (2.3) by points (x_k, y_k) satisfying the equation

$$y^2 = \prod_{i=1}^3 (x - e_i), \quad (2.4)$$

the symplectic form ω can be expressed as

$$\omega = \sum_{i=1}^{N-1} \frac{dx_i}{y_i} \wedge da_i \quad (2.5)$$

Now, locally, we can view X as a fibration over the base \mathbf{C}^{N-1} , with the Abelian variety Σ^{N-1} as fiber. Let the polarization of Σ^{N-1} be denoted by t (t is a two-form which corresponds, physically, to the pairing between the electric and magnetic charges of the $N-1$ $U(1)$ factors). Then the vacua metric (2.1) can be obtained by integrating the $N \times N$ form $t^{N-2} \wedge \omega$ along the fiber.

As the hypermultiplet acquires a non-vanishing mass m , the consistency of the mass formula for BPS states requires that (the cohomology class of) ω vary linearly with m . This is implemented by deforming the above picture (2.2) to configurations φ on Σ admitting a “pole” with residue $m\mu$ at a point P in Σ

$$\overline{\nabla}_A \varphi(z) = m\mu \delta(z, P) \quad (2.6)$$

Here μ is a matrix in the adjoint representation. (Strictly speaking, as we shall see below, this relation can only be implemented after a *singular* gauge transformation on φ . This is as it should be, since φ has a single singularity, and hence must have an essential singularity there.) The condition that this new space still have the same dimension as X forces μ to be a diagonalizable matrix with eigenvalues $1, \dots, 1, -(N-1)$. The family of spectral curves Γ for the theory is then given by the Riemann surfaces of equation

$$R(k, z) \equiv \det(kI - \varphi(z)) = 0. \quad (2.7)$$

where I is the $N \times N$ identity matrix. Donagi and Witten provide an algorithm for writing the equation for Γ under the form

$$P_N + A_2 P_{N-2} + \dots + A_N P_0 = 0 \quad (2.8)$$

where the $P_i = P_i(x, y, k)$ are specific polynomials in x, y, k , monic of degree i in k , with τ -dependent coefficients. The $N-1$ coefficients A_i are identified with the order parameters of the four-dimensional gauge theory [2].

In the formalism of [5][6], the moduli space of vacua of an $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory is to be represented as a leaf in the space $\mathcal{M}_g(n_\alpha, m_\alpha)$ of triples (Γ, E, Q) , where Γ is a Riemann surface of genus g , and E and Q are Abelian integrals

with poles of order at most n_α and m_α respectively at a fixed number N of punctures P_α , $1 \leq \alpha \leq N$. Furthermore, the Seiberg-Witten form $d\lambda$ is represented by the meromorphic one-form $d\lambda = QdE$. In view of the preceding description of Γ , we take the number N of punctures as well as the genus g to be the N of $SU(N)$, n_α to be -1 (i.e., the Abelian differential dE is actually a holomorphic Abelian differential dz on Γ), and Q to be a meromorphic function $Q = k$, with poles of order $m_\alpha = 1$ at each P_α . The moduli space $\mathcal{M}_g(n_\alpha, m_\alpha)$ has dimension [5]

$$\dim \mathcal{M}_g(n_\alpha, m_\alpha) = 5g - 3 + 3N + \sum_{\alpha=1}^N (n_\alpha + m_\alpha) = 8N - 3$$

Let $2\omega_1, 2\omega_2$ be the periods of the torus Σ , (with modulus $\tau = \omega_2/\omega_1$) considered as fixed parameters, and let \mathcal{L} be the lattice $2\omega_1\mathbf{Z} + 2\omega_2\mathbf{Z}$. Then an N -dimensional family of Riemann surfaces Γ with all the properties of (2.7) is given by the following leaf in the canonical foliation of $\mathcal{M}_g(n_\alpha, m_\alpha)$

$$\begin{aligned} \text{Res}_{P_\alpha} dk &= 0, & 1 \leq \alpha \leq N, & & \oint_C dk &= 0 \\ \int_{P_1}^{P_\alpha} dz &\in \mathcal{L}, & 2 \leq \alpha \leq N, & & \oint_C dz &\in \mathcal{L} \\ \text{Res}_{P_\alpha}(k dz) &= -m, & 2 \leq \alpha \leq N, & & \text{Res}_{P_1}(k dz) &= m(N-1). \end{aligned} \tag{2.9}$$

We note that this accounts for $7N - 3$ conditions. The remaining N parameters in $\mathcal{M}_g(n_\alpha, m_\alpha)$, which are thus the parameters for the leaf, are provided by the A_j periods

$$a_j = \frac{1}{2\pi i} \oint_{A_j} d\lambda,$$

where A_j, B_j are a canonical homology basis on Γ . We observe that there are N of them, and thus one more than the dimension of the moduli space of vacua of the $SU(N)$ gauge theory. This is as it should be, since the $SU(N)$ traceless condition $\sum_{i=1}^N a_i = 0$ is yet to be imposed. However, the above set-up is clearly invariant under shifts of the meromorphic function k by a constant, and it is this arbitrariness which allows us to impose the traceless condition.

Remarkably, as anticipated by Martinec [4] and established by Krichever, the spectral curves specified by (2.7) and (2.9) are precisely the spectral curves written down much earlier by Krichever [3] for the elliptic Calogero-Moser system

$$p_i = \dot{x}_i, \quad \dot{p}_i = m^2 \sum_{j \neq i} \wp'(x_i - x_j), \quad 1 \leq i, j \leq N, \tag{2.10}$$

where $\wp(z)$ is the Weierstrass \wp -function with periods $2\omega_1, 2\omega_2$. In fact, it was shown by Krichever in [3] that (2.10) is equivalent to the Lax equation $\dot{L} = [M, L]$, where $(L(z), M(z))$ is the following Lax pair with spectral parameter $z \in \Sigma$

$$\begin{aligned} L_{ij}(z) &= p_i \delta_{ij} - m(1 - \delta_{ij})\Phi(x_i - x_j, z) \\ M_{ij}(z) &= m\delta_{ij} \sum_{k \neq i} \wp(x_i - x_k) + m(1 - \delta_{ij})\Phi'(x_i - x_j, z) \end{aligned} \quad (2.11)$$

Here $\Phi(x, z)$ is the function defined by

$$\Phi(x, z) \equiv \frac{\sigma(z-x)}{\sigma(z)\sigma(x)} e^{\zeta(z)x} \quad (2.12)$$

where $\sigma(z)$ and $\zeta(z)$ are the Weierstrass functions, given by

$$\sigma(z) = z \prod_{n_1, n_2} \left(1 - \frac{z}{n_1\omega_1 + n_2\omega_2}\right) \exp\left(\frac{z}{n_1\omega_1 + n_2\omega_2} + \frac{1}{2}\left(\frac{z}{n_1\omega_1 + n_2\omega_2}\right)^2\right),$$

and $\zeta(z) = \partial_z \log \sigma(z)$. Recall that we have $\wp(z) = -\partial_z \zeta(z)$ and $\sigma(z) = z + \mathcal{O}(z^5)$. Also, $\Phi'(x, z)$ denotes the x -derivative of $\Phi(x, z)$, and the key identity on the function $\Phi(x, z)$ used in showing that (2.10) follows from the Lax equation is

$$\Phi(x, z)\Phi'(y, z) - \Phi(y, z)\Phi'(x, z) = (\wp(x) - \wp(y))\Phi(x + y, z).$$

The functions $\Phi(x, z)$ and $L(z)$ are doubly periodic in z with periods $2\omega_1$ and $2\omega_2$ and thus well defined on the torus Σ . $L(z)$ has an essential singularity at $z = 0$, of the form

$$L_{ij}(z) \sim m(1 - \delta_{ij}) \frac{e^{-\frac{1}{z}(x_i - x_j)}}{z} \quad (2.13)$$

However, under the local and singular gauge transformation $L \rightarrow \tilde{L} = G^{-1}L(z)G$, with $G_{ij} = \delta_{ij}e^{\zeta(z)x_i}$, L is transformed into

$$\tilde{L}_{ij}(z) = p_i \delta_{ij} - m(1 - \delta_{ij}) \frac{\sigma(z - x_i + x_j)}{\sigma(z)\sigma(x_i - x_j)} \quad (2.14)$$

which is meromorphic near $z = 0$. Thus the equation of the spectral curve Γ for the elliptic Calogero-Moser system

$$R(k, z) = \det(kI - L(z)) = 0 \quad (2.15)$$

is doubly periodic as well as meromorphic in z . The invariance of the theory under shifts of k by constants corresponds in this set-up to the freedom of adding a constant matrix to

$L(z)$, or equivalently, to shift all momenta p_i . The spectral curve Γ is a N -sheeted covering of the torus Σ . The holomorphic Abelian differential dz on Σ pulls back to a holomorphic Abelian differential on Γ , which we still denote by dz . The solution k of (2.15) defines a single-valued meromorphic function on Γ . Clearly, all its poles P_1, \dots, P_N lie on top of the pole at $z = 0$ of $R(k, z)$. This shows that the second equation in (2.9) must be satisfied. It is also easy to verify all the other constraints in (2.9).

III. THE CLASSICAL ORDER PARAMETERS

In the case of $\mathcal{N} = 2$ supersymmetric gauge theories with classical gauge groups, the classical order parameters of the theory are apparent from the equation of the spectral curve. For example, for $SU(N)$ theories with matter in the fundamental representation, they are recognizable as the parameters \bar{a}_k in the equation [7]

$$y^2 = \prod_{i=1}^N (x - \bar{a}_k)^2 - \Lambda^{2N_c - N_f} \prod_{j=1}^{N_f} (x + m_j) \quad (3.1)$$

The issue is more subtle in the adjoint representation case. Presumably the coefficients A_k of the equation (2.8) are gauge invariant polynomials in the scalar field ϕ [2], although this is not entirely manifest. In the Calogero-Moser formalism, we have $2N$ parameters x_i, p_i which are natural from the dynamical system but not from the gauge theory viewpoint. We can also try to characterize the polynomial $R(k, z)$ by its coefficients, which are elliptic functions and hence linear combinations of the Weierstrass \wp -function and its derivatives. However, the number of these terms exceeds N , and linear constraints have to be imposed on them [14], which make their gauge theoretic interpretation obscure. Thus a first basic step of our approach is a more appropriate parametrization for (2.8)(2.9)(2.15), which will shed light on the correct order parameters.

To achieve this, we begin by reducing the singularity at $z = 0$ in $R(k, z)$ to a simple pole by performing a shift in k . This can be done at the cost of replacing the double periodicity of $R(k, z)$ by a more complicated transformation law. We define the function $h_n(z)$ by

$$h_n(z) \equiv \frac{1}{\vartheta_1(\frac{z}{2\omega_1}|\tau)} \frac{\partial^n}{\partial z^n} \vartheta_1(\frac{z}{2\omega_1}|\tau) \quad (3.2)$$

where the Jacobi ϑ -function is given by

$$\vartheta_1(u|\tau) = \sum_{r \in \frac{1}{2} + \mathbf{Z}} e^{i\pi r^2 \tau + 2i\pi r(u + \frac{1}{2})} \quad (3.3a)$$

In view of the transformation laws of the ϑ -function under shifts of u

$$\begin{aligned}\vartheta_1(u+1|\tau) &= e^{i\pi}\vartheta_1(u|\tau) \\ \vartheta_1(u+\tau|\tau) &= e^{-\pi i\tau-2\pi i(u-\frac{1}{2})}\vartheta_1(u|\tau),\end{aligned}\tag{3.3b}$$

we see that the functions $h_n(z)$ transform as follows

$$\begin{aligned}h_n(z+2\omega_1) &= h_n(z) \\ h_n(z+2\omega_2) &= \sum_{p=0}^n \binom{n}{p} \beta^{n-p} h_p(z) \quad \beta = -\frac{i\pi}{\omega_1}\end{aligned}\tag{3.4}$$

In particular, we have $h_1(z+2\omega_2) = h_1(z) + \beta$, since $h_0 = 1$.

We now introduce the function $f(k, z)$ by

$$f(k, z) \equiv R(k - mh_1(z), z)\tag{3.5}$$

While $R(k, z)$ has a pole of order N in z at $z = 0$ and is doubly periodic in z , the function $f(k, z)$ is constructed to have only a simple pole at $z = 0$, at the cost of the following non-trivial transformation laws under shifts of z

$$\begin{aligned}f(k, z+2\omega_1) &= f(k, z) \\ f(k, z+2\omega_2) &= f(k - \beta m, z).\end{aligned}\tag{3.6}$$

We claim that $f(k, z)$ can be simply expressed in terms of a single polynomial of degree N whose zeroes are linearly related to the classical order parameters of the $\mathcal{N} = 2$ supersymmetric gauge theory in the Coulomb phase. This can be seen as follows. The residue at the pole $z = 0$ is a polynomial in k of degree N , so that the residues at the poles $z = 2\omega_1\mathbf{Z} + 2\omega_2\mathbf{Z}$ are shifts of the polynomial at $z = 0$, according to the monodromy law (3.6). Thus, the pole structure of $f(k, z)$ is completely determined by a polynomial in k of degree N . The relation with the classical order parameters will be established below.

To derive a concrete relation, we make use of (3.6) to derive a transformation law for the coefficients $f_n(z)$ in $f(k, z) = \sum_{n=0}^N f_n(z) k^{N-n}$

$$\begin{aligned}f_n(z+2\omega_1) &= f_n(z) \\ f_n(z+2\omega_2) &= \sum_{p=0}^n f_p(z) \binom{N-p}{n-p} (-\beta m)^{n-p}\end{aligned}\tag{3.7}$$

We notice that the transformation laws in (3.7) and (3.4) are closely related. One can show that the functions $f_n(z)$ may be expressed as linear combinations of the functions $h_p(z)$ with $0 \leq p \leq n$. To do so, one constructs a linear combination $\tilde{f}_n(z)$ (with β -dependent coefficients) of $h_p(z)$ with $0 \leq p \leq n$ that has the same transformation laws (3.7) and the same simple pole at $z = 0$ as $f_n(z)$. The difference $f_n(z) - \tilde{f}_n(z)$ is then holomorphic and doubly periodic and thus must be constant. The constant may be absorbed in the $h_0(z) = 1$ term in $\tilde{f}_n(z)$.

Thus, $f(k, z)$ admits a decomposition under the form

$$f(k, z) = \sum_{n=0}^N h_n(z) Q_{N-n}(k) \quad (3.8)$$

with $Q_p(k)$ a polynomial in k of degree p . In view of (3.6) the $Q_p(k)$ satisfy the following recursion relation

$$Q_p(k - \beta m) = \sum_{n=0}^p \binom{N-n}{p-n} \beta^{p-n} Q_n(k). \quad (3.9)$$

In terms of the generating function $H(t, k) = \sum_{p=0}^N t^{N-p} Q_p(k)$, (3.9) is equivalent to $H(t + \beta, k + \beta m) = H(t, k)$. Since $H(t, k)$ is polynomial both in t and in k , this condition requires that $H(t, k) = H(0, k - tm)$ depends only upon the combination $k - tm$. Defining the polynomial of a single variable $H(k) \equiv H(0, k)$, we have

$$H(t, k) = H(k - tm) = \sum_{p=0}^N \frac{(-)^p (tm)^p}{p!} H^{(p)}(k), \quad (3.10)$$

which identifies the polynomials $Q_n(k)$ of (3.7) as

$$Q_{N-n}(k) = \frac{(-m)^n}{n!} H^{(n)}(k) \quad (3.11)$$

In summary, we have shown that the function $f(k, z)$ can be re-expressed in terms of a polynomial $H(k)$ of degree N . The coefficient of the leading monomial is given by the general form of $f(k, z)$ and equals 1. The remaining N coefficients, or equivalently the N roots of the polynomial $H(k)$, represent the classical order parameters of the $\mathcal{N} = 2$ supersymmetric gauge theory. In the next section, we shall demonstrate this fact explicitly by calculating the classical limit ($\text{Im } \tau \rightarrow \infty$) of the quantum order parameters.

Combining (3.2), (3.8) and (3.11), we find a simple form for $f(k, z)$ in terms of $H(k)$:

$$f(k, z) = \frac{1}{\vartheta_1(\frac{z}{2\omega_1}|\tau)} \sum_{n=0}^N \frac{1}{n!} \frac{\partial^n}{\partial z^n} \vartheta_1(\frac{z}{2\omega_1}|\tau) \left(-m \frac{\partial}{\partial k}\right)^n H(k) \quad (3.12)$$

or, even more succinctly,

$$f(k, z) = \frac{1}{\vartheta_1(\frac{z}{2\omega_1}|\tau)} \vartheta_1\left(\frac{1}{2\omega_1}\left\{z - m\frac{\partial}{\partial k}\right\}|\tau\right) H(k). \quad (3.13)$$

The original equation for the curve Γ was given by $R(\tilde{k}, z) = 0$, and the associated Seiberg-Witten differential is then $d\lambda = \tilde{k}dz$. It is convenient to translate these expressions in terms of variables that are naturally adapted to the function $f(k, z)$ instead. This is achieved by setting $k \equiv \tilde{k} + mh_1(z) + \frac{1}{2}\beta m$, so that the equation for f and the Seiberg-Witten differential in terms of k become

$$\begin{aligned} 0 &= f\left(k - \frac{1}{2}\beta m, z\right) \\ d\lambda &= \tilde{k}dz = kdz - mh_1(z)dz - \frac{1}{2}\beta m dz \end{aligned} \quad (3.14)$$

It is often convenient to ignore the term $-\frac{1}{2}\beta m dz$ in $d\lambda$, as we shall do in the beginning of Section IV.(c). This term contributes to a_i and a_{Di} a constant i -independent shift, whose effect on the prepotential is easily read off (c.f. (4.37) and (4.38) below). The simple poles at the N lifts of the pole at $z = 0$ are clearly exhibited by this expression in the part $-mh_1(z)dz$, with a remaining pole on just a single sheet left in the part kdz . The residue at this remaining pole is readily checked to be Nm , in agreement with the residue conditions of (2.9). (Notice that we could have chosen the period $2\omega_1 = -2\pi i$, so that $\beta = 1$; we shall instead keep track of general β throughout and show as a check that physical quantities, such as the prepotential, are independent of β , when expressed in terms of the quantum order parameters.)

Using the series expansion for ϑ -functions, given by (3.3a), we find that the equation for the spectral curve $f(k - \frac{1}{2}\beta m, z) = 0$ admits the following series expansion in powers of $q = e^{2i\pi\tau}$

$$\sum_{n \in \mathbf{Z}} (-)^n q^{\frac{1}{2}n(n-1)} e^{\beta n z} H(k - \beta m n) = 0. \quad (3.15)$$

This series is remarkably sparse : the orders to which corrections occur grow quadratically !

IV. THE QUANTUM ORDER PARAMETERS AND THE EFFECTIVE PREPOTENTIAL

We are now ready to evaluate the quantum order parameters a_i , their duals a_{Di} and the prepotential $\mathcal{F}(a_i, \tau)$, given by

$$a_i = \frac{1}{2\pi i} \oint_{A_i} d\lambda, \quad a_{Di} = \frac{1}{2\pi i} \oint_{B_i} d\lambda, \quad a_{Di} = \frac{\partial \mathcal{F}}{\partial a_i}, \quad (4.1)$$

in the weak coupling limit. Recall that the coupling τ is related to the gauge coupling e and θ -angle of the microscopic gauge theory by

$$q = e^{2i\pi\tau} = e^{-\frac{8\pi^2}{e^2} + i\theta}.$$

Thus the weak-coupling limit corresponds to vanishing q . Our main goals are

- (a) to justify the zeroes of the polynomial $H(k)$ introduced in the previous section as classical order parameters;
- (b) to establish that the spectral curves for the elliptic Calogero-Moser system are the correct Seiberg-Witten spectral curves for the $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory with matter in the adjoint representation. In particular, the prepotential \mathcal{F} of (4.1) must exhibit the correct logarithmic singularities which would arise from the one-loop perturbative effects of the supersymmetric gauge theory;
- (c) to evaluate the contributions to \mathcal{F} of instanton processes. In this section, we shall carry this out directly from (4.1) to one instanton order. In the next, we shall derive a renormalization group equation for \mathcal{F} from which the one- and two-instanton contributions can be read off at once. The two independent methods also serve as mutual checks for each other.

Before giving the details of the calculation, it may be helpful to discuss some aspects of our method and of the underlying geometry.

- The Riemann surface Γ defined by the equation (2.9) is an N -sheeted covering of the torus Σ . If we represent Σ by the lattice $2\omega_1\mathbf{Z} + 2\omega_2\mathbf{Z}$ and let A and B be the usual canonical homology cycles on Σ , then a homology basis (A_i, B_i) for Γ is just given by the lifts A_i, B_i to each sheet of the A, B cycle on the base Σ .

- At the classical limit $q = 0$, the cycle A shrinks to a point, and the base Σ degenerates into a sphere with two marked punctures (to be identified with the shrunken cycle A). If we set

$$w = e^{\beta z}, \tag{4.2}$$

the two marked punctures are given by $w = 0$ and $w = \infty$. Upstairs, the cycles A_i also degenerate into $2N$ punctures, with a pair on each sheet. If we also view each sheet as a sphere with two marked punctures, then the shrunken A_i corresponds to two punctures k_i and $k_i + \beta m$, which lie respectively over the punctures $w = 0$ and $w = \infty$ on the base. We shall show that the punctures k_i are precisely the zeroes of the polynomial $H(k)$

$$H(k) = \prod_{i=1}^N (k - k_i) \equiv (k - k_i) H_i(k) \tag{4.3}$$

In (4.3), we have taken the opportunity to define the polynomials $H_i(k)$, which we shall use shortly.

- As q moves away from 0, each of the punctures in both the base Σ and the covering Γ opens into a cut, the edges of which reconstitute the A cycle downstairs and the A_i cycles upstairs upon regluing. Let the cut near k_i run from a point k_i^- to another point k_i^+ . The points k_i^\pm are identified as turning points, i.e., solutions of the equation

$$\frac{dk}{dz} = 0 \quad (4.4)$$

which we can solve perturbatively in q . They lie respectively over the end points w^\pm of the cut downstairs in Σ . Similarly, we have cuts at the other end going from $k_i^- + \beta m$ to $k_i^+ + \beta m$. The B_i cycle in Γ can be represented by a path on the i -th sheet going from $k_i^+ + \beta m$ to k_i^+ . This is the path we shall use for evaluating a_{Di} . For a_i , we shall use any simple closed path surrounding the cut near k_i in the clockwise direction. With these conventions, the A_i and B_i cycles have canonical intersection form $\#(A_i, B_j) = \delta_{ij}$. The evaluation of a_i is much simpler than that of a_{Di} , since perturbatively in q , the cut shrinks to a point, and the residue formula applies.

- Our calculations will be carried out on each sheet upstairs rather than downstairs, i.e., in terms of the variable k rather than the variable z . This is suggested by the role of k_i as classical order parameters, and is possible since the cycles A_i and B_i both lie on a single copy of the complex plane. For this however, we need a careful expansion of the differential dz in terms of dk , which is derived in (4.13)(4.16) and (4.19) below. Our starting point is the equation $f(k, z) = 0$ of (3.15), which we shall re-express in terms of the variable w :

$$H(k) - wH(k - \beta m) + \sum_{n=1}^{\infty} (-)^n q^{\frac{1}{2}n(n+1)} [w^{-n} H(k + \beta mn) - w^{n+1} H(k - \beta m - \beta mn)] = 0. \quad (4.5)$$

We note that at the degeneration point $q = 0$, the zeroes k_i of $H(k)$ and $k_i + \beta m$ of $H(k - \beta m)$ correspond indeed respectively to the nodes $e^{\beta z} = w = 0$ and $e^{\beta z} = w = \infty$ of the degenerating torus Σ .

We now carry out the above program, up to one-instanton order, that is, up to and including $\mathcal{O}(q)$ terms.

(a) The Turning Points k_i^\pm

The turning points k_i^\pm are obtained by imposing the constraint $dk/dw = 0$ together with the equation $f(k - \frac{1}{2}\beta m, z) = 0$ of the curve. Up to and including first order in q , they are then solutions of the system

$$\begin{aligned} H(k) - wH(k - \beta m) - \frac{q}{w}H(k + \beta m) + qw^2H(k - 2\beta m) &= 0 \\ -H(k - \beta m) + \frac{q}{w^2}H(k + \beta m) + 2qwH(k - 2\beta m) &= 0 \end{aligned} \quad (4.6)$$

Solving for $H(k)$ and $H(k - \beta m)$, this system can be put under the form

$$\begin{aligned} H(k - \beta m) &= \frac{q}{w^2}H(k + \beta m) + 2qwH(k - 2\beta m) \\ H(k) &= 2\frac{q}{w}H(k + \beta m) + qw^2H(k - 2\beta m) \end{aligned} \quad (4.7)$$

Near the solution $w = 0$ for $q = 0$, we note that $w \sim q^{\frac{1}{2}}$ so that the terms suppressed by a power of w^3 can be ignored. Eliminating w between the remaining terms yields an equation for k

$$H(k)^2 = 4qH(k + \beta m)H(k - \beta m) \quad (4.8)$$

With the help of the function $H_i(k)$, defined in (4.3), we rewrite (4.8) as an iterative equation for k :

$$(k - k_i)^2 = 4q \frac{H(k + \beta m)H(k - \beta m)}{H_i(k)^2} \quad (4.9)$$

This equation admits two solutions, k_i^\pm , which correspond to the two end points of the branch cut associated with the A_i -cycle on Γ . We shall need their explicit forms up to order $\mathcal{O}(q)$, and also present the associated value of w , to be made use of later on :

$$\begin{aligned} k_i^\pm &= k_i \pm q^{\frac{1}{2}}k_i^{(1)} + qk_i^{(2)}, \\ w(k_i^\pm) &= 2qH(k_i^\pm + \beta m)H(k_i^\pm)^{-1}. \end{aligned} \quad (4.10)$$

Here, we have introduced the following q -independent functions

$$\begin{aligned} k_i^{(1)} &= 2 \frac{H^{\frac{1}{2}}(k_i - \beta m)H^{\frac{1}{2}}(k_i + \beta m)}{H_i(k_i)} \\ k_i^{(2)} &= 2 \frac{d}{dk} \left(\frac{H(k - \beta m)H(k + \beta m)}{H_i(k)^2} \right) \Big|_{k=k_i} \end{aligned} \quad (4.11)$$

Near the solution $w = \infty$ of the degenerate case $q = 0$, we note that $w \sim q^{-\frac{1}{2}}$, so that the terms suppressed by a power of w^{-3} may be ignored in (4.6). Proceeding as before, we find that the turning points occur precisely at $k_i^\pm + \beta m$ with

$$w(k_i^\pm + \beta m) = \frac{1}{2q}H(k_i^\pm)H(k_i^\pm - \beta m)^{-1} = \frac{1}{q}w(k_i^\pm) \quad (4.12)$$

where the last equality is as expected from the definition of $w = e^{\beta z}$.

(b) Expansion of the Differential dz

Our next step is to rewrite the variable z (equivalently $w = e^{\beta z}$) in terms of the variable k on each sheet. It is convenient to introduce the new variable y by

$$w = y \frac{H(k)}{H(k - \beta m)}. \quad (4.13)$$

In terms of y , the equation (4.5) can be rewritten as a fixed point equation

$$y = 1 + qF(y) \quad (4.14)$$

where the function $F(y)$ is defined by

$$F(y) = \sum_{n=1}^{\infty} q^{\frac{1}{2}n(n+1)-1} (-)^n [y^{-n} \eta_n(k, \beta) - y^{n+1} \eta_n(k - \beta m, -\beta)]$$

$$\eta_n(k, \beta) = \frac{H(k + \beta m n) H(k - \beta m)^n}{H(k)^{n+1}}. \quad (4.15)$$

Formally, an iterative solution to all orders is given by (c.f. [7])

$$y = 1 + \sum_{n=1}^{\infty} q^n y_n, \quad y_n = \frac{1}{n!} \frac{\partial^{n-1}}{\partial y^{n-1}} F^n(y) \Big|_{y=1}. \quad (4.16)$$

Keeping only the expansion terms up to order q will yield

$$y = 1 - q\eta_1(k, \beta) + q\eta_1(k - \beta m, -\beta)$$

This approximation is valid as long as neither η_1 grows too fast as $q \rightarrow 0$. In particular, it is applicable in the evaluation of the A_i periods, since the integration contours can be chosen to be at a finite fixed distance from the cuts and the points k_i in this case.

However, in the evaluation of the B_i periods, the points on the cuts can come within a distance $q^{\frac{1}{2}}$ of the nodes k_i and $k_i + \beta m$. This invalidates in this case the above naive approximation, and requires a more careful analysis. To order up to and including $\mathcal{O}(q)$ we can write the equation (4.15-16) as

$$y = 1 - y^{-1} q\eta_1(k, \beta) + y^2 q\eta_1(k - \beta m, -\beta) \quad (4.17)$$

A key observation is that $\eta_1(k, \beta) = \infty$ is equivalent to $\eta_1(k - \beta m, -\beta) = 0$ and vice versa. Thus the functions $y = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4q\eta_1(k, \beta)}$ and $y^{-1} = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4q\eta_1(k - \beta m, -\beta)}$

are good approximations of the solution near k_i and $k_i + \beta m$ respectively. Since they each approach 1 near the other solution, the natural candidate for a solution valid everywhere is

$$y = (1 + \sqrt{1 - 4q\eta_1(k, \beta)})(1 + \sqrt{1 - 4q\eta_1(k - \beta m, -\beta)})^{-1} \quad (4.18)$$

This can indeed be checked to be a solution of (4.17) up to and including order $\mathcal{O}(q)$.

We make use of the following expansion

$$\log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}\right) = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \frac{1}{n} x^n.$$

Rewriting $\eta_1(k, \beta)$ and $\eta_1(k - \beta m, -\beta) = \eta_1(k - \beta m, \beta)$ in terms of $k_i^{(1)}$ and $k_i^{(2)}$, we obtain the following pole expansion

$$(4\eta_1(k, \beta))^n = \sum_{i=1}^N \left\{ \frac{1}{(k - k_i)^{2n}} (k_i^{(1)})^{2n} + \frac{1}{(k - k_i)^{2n-1}} 2n (k_i^{(1)})^{2n-2} k_i^{(2)} + \dots \right\}$$

(where we have ignored poles of order $2n - 2$ and lower). This results in the following expansion for $\log y$ to this order

$$\begin{aligned} \log y = & -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \frac{1}{n} q^n \sum_{i=1}^N (k_i^{(1)})^{2n} \left[\frac{1}{(k - k_i)^{2n}} \right. \\ & \left. + 2n \frac{k_i^{(2)}}{(k_i^{(1)})^2} \frac{1}{(k - k_i)^{2n-1}} + \dots - \{k \rightarrow k - \beta m\} \right] \end{aligned} \quad (4.19)$$

We note that, as in the case of matter in the fundamental representation treated in [7], we need to keep in this expansion all powers q^n of q . This is because the factors $(k - k_i)^{-2n}$ and $(k - k_i)^{-2n+1}$ can be as large as q^{-n} and $q^{-n+\frac{1}{2}}$. Together with (4.13), this provides us with the desired expansion for $dz = \frac{1}{\beta} d\log w$.

(c) Evaluation of the Periods a_i and a_{Di}

Since the expansion (4.19) for the Seiberg-Witten differential $d\lambda$ is an expansion in terms of poles, the methods of [7] apply to give expansions for a_i and a_{Di} . Following our discussion at the beginning of this Section, we shall reexpress the Seiberg-Witten differential $d\lambda$ of (3.14) in terms of the variable k . As suggested after the equation (3.14), we can momentarily ignore the term $-\frac{1}{2}\beta m dz$, and restore it only at the end, in the final formulas for the order parameters and the prepotential. To lighten the notation, we shall still use in this Section IV.(c) the notation $d\lambda$, a_i , and a_{Di} as if no change had been made.

Only in Section IV.(d), when an explicit distinction has to be made between the original case and the case where $-\frac{1}{2}\beta m$ has been dropped, do we need to introduce a new notation $d\tilde{\lambda}$, \tilde{a}_i , and \tilde{a}_{Di} for the Seiberg-Witten differential and quantum order parameters in the latter case. We have then

$$\begin{aligned} d\lambda &= \frac{1}{\beta} k d \log w(k) - m h_1(z) dz \\ &= \frac{1}{\beta} \left(k d \log \frac{H(k)}{H(k - \beta m)} - (\log y) dk + d(k \log y) \right) - m h_1(z) dz \\ &= \frac{1}{\beta} \left(d\lambda^{(0)} + d\lambda^{(1)} + d\lambda^{(2)} + d(k \log y) \right) - m h_1(z) dz \end{aligned} \quad (4.20)$$

where we have labelled the term $k d \log (H(k)/H(k - \beta m))$ by $d\lambda^{(0)}$, and separated the contributions of the series (4.19) to the term $-(\log y) dk$ into the contribution $d\lambda^{(1)}$ for $n = 1$ and the contribution $d\lambda^{(2)}$ for $n \geq 2$.

The A_i periods yield the quantum order parameters a_i and are readily evaluated to order $\mathcal{O}(q)$

$$a_i = \frac{1}{\beta} (k_i + \frac{1}{2} q k_i^{(2)}). \quad (4.21)$$

This confirms that the parameters k_i of our parametrization (3.14)(4.3) are indeed the classical order parameters, i.e., the limits as $q \rightarrow 0$ of the quantum order parameters a_i given by the A_i -periods of the Seiberg-Witten differential.

We turn now to the B_i periods a_{Di} , which we break up in parallel with (4.20), with contributions $a_{Di}^{(p)}$ arising from the integration of the differentials $d\lambda^{(p)}$, $p = 0, 1, 2$.

$$\begin{aligned} -2\pi i \beta a_{Di} &= \int_{k_i^+}^{k_i^+ + \beta m} \beta d\lambda \\ &= a_{Di}^{(0)} + a_{Di}^{(1)} + a_{Di}^{(2)} + \int_{k_i^+}^{k_i^+ + \beta m} \{d(k \log y) - \beta m h_1(z) dz\} \end{aligned} \quad (4.22)$$

Substituting (4.19) in, all the integrals are easily evaluated. We find

$$\begin{aligned} a_{Di}^{(0)} &= \sum_{j=1}^N \left\{ (k_j - \beta m) \log \frac{k_i^+ - k_j + \beta m}{k_i^+ - k_j} + (k_j + \beta m) \log \frac{k_i^+ - k_j - \beta m}{k_i^+ - k_j} \right. \\ &\quad \left. + \beta m \log \frac{k_i^+ - k_j + \beta m}{k_i^+ - k_j} \right\} \\ a_{Di}^{(1)} &= \frac{1}{4} q \sum_{j=1}^N \left[(k_j^{(1)})^2 \left(\frac{2}{k_i^+ - k_j} - \frac{1}{k_i^+ - k_j + \beta m} - \frac{1}{k_i^+ - k_j - \beta m} \right) \right] \end{aligned}$$

$$\begin{aligned}
& + 2k_j^{(2)} \log \frac{k_i^+ - k_j + \beta m}{k_i^+ - k_j} + 2k_j^{(2)} \log \frac{k_i^+ - k_j - \beta m}{k_i^+ - k_j} \Big] \\
a_{Di}^{(2)} = & \sum_{n=2}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \frac{1}{2n} q^n \left[\frac{2}{2n-1} \frac{(k_i^{(1)})^{2n}}{(k_i^+ - k_i)^{2n-1}} + \frac{4n}{2n-2} \frac{k_i^{(2)} (k_i^{(1)})^{2n-2}}{(k_i^+ - k_i)^{2n-2}} \right]
\end{aligned} \tag{4.23}$$

The $a_{Di}^{(2)}$ terms can be simplified, using the expansion (4.10) for k_i^+ , and the numerical series of [7]

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \frac{1}{n(2n-1)} &= \frac{3}{2} - 2\log 2 \\
\sum_{n=2}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \frac{1}{n(2n-2)} &= 1 - \log 2.
\end{aligned}$$

One finds

$$a_{Di}^{(2)} = \left(\frac{3}{2} - 2\log 2\right) q^{\frac{1}{2}} k_i^{(1)} + (1 - \log 2) q k_i^{(2)} \tag{4.24}$$

Next, using the second expression in (4.10), we recognize the third term in $a_{Di}^{(0)}$ as

$$\begin{aligned}
\beta m \sum_{j=1}^N \log \frac{k_i^+ - k_j + \beta m}{k_i^+ - k_j} &= \beta m \log \frac{H(k_i^+ + \beta m)}{H(k_i^+)} \\
&= \beta m (\log w(k_i^+) - \log q - \log 2)
\end{aligned} \tag{4.25}$$

Using (4.10) and (4.12), as well as the definition of the variable y in (4.14), we see that $y(k_i^+) = \frac{1}{2}$, while $y(k_i^+ + \beta m) = 2$. With these results, we may now easily evaluate the following integral

$$\begin{aligned}
\int_{k_i^+}^{k_i^+ + \beta m} d(k \log y) &= (k_i^+ + \beta m) \log y(k_i^+ + \beta m) - k_i^+ \log y(k_i^+) \\
&= (2k_i^+ + \beta m) \log 2.
\end{aligned} \tag{4.26}$$

Using the definition of $h_1(z)$ in (3.2), we evaluate the integral

$$-\beta m \int_{k_i^+}^{k_i^+ + \beta m} h_1(z) dz = i\pi \beta m \tau - \beta m \log w(k_i^+) \tag{4.27}$$

Combining these contributions, and regrouping the logarithmic terms in $a_{Di}^{(0)}$ and $a_{Di}^{(1)}$ using (4.21), we find

$$\begin{aligned}
-2\pi i \beta a_{Di} &= \sum_{j=1}^N \beta (a_j - m) \log \left(\frac{k_i^+ - k_j + \beta m}{k_i^+ - k_j} \right) + \beta (a_j + m) \log \left(\frac{k_i^+ - k_j - \beta m}{k_i^+ - k_j} \right) \\
&+ \frac{1}{4} q \sum_{j=1}^N \left(\frac{2(k_j^{(1)})^2}{k_i^+ - k_j} - \frac{(k_j^{(1)})^2}{k_i^+ - k_j + \beta m} - \frac{(k_j^{(1)})^2}{k_i^+ - k_j - \beta m} \right) \\
&+ \frac{3}{2} q^{\frac{1}{2}} k_i^{(1)} + q k_i^{(2)} + 2\beta \log 2 a_i - \pi i \beta m \tau
\end{aligned} \tag{4.28}$$

Next, we evaluate (4.8) on $k = k_i^+$. By taking the logarithm, it follows that k_i^+ must satisfy the following identity

$$0 = \sum_{j=1}^N \left\{ \log \left(\frac{k_i^+ - k_j + \beta m}{k_i^+ - k_j} \right) + \log \left(\frac{k_i^+ - k_j - \beta m}{k_i^+ - k_j} \right) \right\} + \log q + 2 \log 2. \quad (4.29)$$

Dividing all of equation (4.28) by a factor of $-\beta$ and adding (4.29) multiplied by a factor of a_i to (4.28), allows us to recast (4.28) as

$$\begin{aligned} 2\pi i a_{Di} = & a_i \log q - \frac{3}{2\beta} q^{\frac{1}{2}} k_i^{(1)} - \frac{1}{\beta} q k_i^{(2)} + \pi i m \tau \\ & - \sum_{j=1}^N \left\{ (a_j - a_i - m) \log \left(\frac{k_i^+ - k_j + \beta m}{k_i^+ - k_j} \right) + (a_j - a_i + m) \log \left(\frac{k_i^+ - k_j - \beta m}{k_i^+ - k_j} \right) \right\} \\ & - \frac{1}{4\beta} q \sum_{j=1}^N \left(\frac{2(k_j^{(1)})^2}{k_i^+ - k_j} - \frac{(k_j^{(1)})^2}{k_i^+ - k_j + \beta m} - \frac{(k_j^{(1)})^2}{k_i^+ - k_j - \beta m} \right) \end{aligned} \quad (4.30)$$

It remains to express all terms solely as a function of the quantum order parameters a_i . This is achieved by expanding k_i^+ according to (4.10) and recombining according to (4.21). The result may be simplified with the help of the following two identities, valid up to $\mathcal{O}(q)$

$$- \sum_{j \neq i} \frac{1}{a_i - a_j} + \frac{1}{2} \sum_{j=1}^N \left[\frac{1}{a_i - a_j + m} + \frac{1}{a_i - a_j - m} \right] = \beta \frac{k_i^{(2)}}{(k_i^{(1)})^2} \quad (4.31)$$

and

$$\begin{aligned} \frac{\partial}{\partial a_i} \sum_{j \neq i} (k_j^{(1)})^2 &= \frac{\partial}{\partial a_i} 4\beta^2 \sum_{j \neq i} \frac{\prod_{l=1}^N (a_j - a_l + m)(a_j - a_l - m)}{\prod_{l \neq j} (a_j - a_l)^2} \\ &= - \sum_{j \neq i} \left(\frac{(k_j^{(1)})^2}{a_i - a_j + m} + \frac{(k_j^{(1)})^2}{a_i - a_j - m} - 2 \frac{(k_j^{(1)})^2}{a_j - a_i} \right) \end{aligned} \quad (4.32)$$

We can now combine the equations (4.30), (4.31), and (4.32), and obtain

$$\begin{aligned} 2\pi i a_{Di} = & a_i \log q + \frac{1}{2} m \log q \\ & - \sum_{j=1}^N \left[2(a_i - a_j) \log(a_i - a_j) - (a_i - a_j + m) \log(a_i - a_j + m) \right. \\ & \left. - (a_i - a_j - m) \log(a_i - a_j - m) \right] + \frac{1}{4\beta^2} q \frac{\partial}{\partial a_i} \sum_{j=1}^N (k_j^{(1)})^2 \end{aligned} \quad (4.33)$$

We note that to order $\mathcal{O}(q)$, we may replace k_i^+ by βa_i in the expression (4.11) for $k_i^{(1)}$. Thus we arrive at the following final formula for a_{Di} expressed entirely in terms of the quantum order parameters a_i

$$\begin{aligned}
2\pi i a_{Di} = & a_i \log q + \frac{1}{2} m \log q - \sum_{j=1}^N [2(a_i - a_j) \log(a_i - a_j) \\
& - (a_i - a_j + m) \log(a_i - a_j + m) - (a_i - a_j - m) \log(a_i - a_j - m)] \\
& - q m^2 \frac{\partial}{\partial a_i} \sum_{j=1}^N \prod_{l \neq j} \left(1 - \frac{m^2}{(a_l - a_j)^2}\right).
\end{aligned} \tag{4.34}$$

Notice that the dependence on the parameter β , which is related to the A -period of Σ and which can be chosen at will, has completely disappeared from the above expression, as expected.

(d) The Prepotential

The expression for the prepotential is obtained by integration, and may be separated into a classical part $\mathcal{F}^{\text{class}}$, a perturbative part $\mathcal{F}^{\text{pert}}$, which arises only from one loop order, and an n -instanton part $\mathcal{F}^{(n)}$.

$$\mathcal{F} = \mathcal{F}^{\text{class}} + \mathcal{F}^{\text{pert}} + \sum_{n=1}^{\infty} \mathcal{F}^{(n)} \tag{4.35}$$

We need at this point to make a distinction between the prepotential \mathcal{F} , as determined by the original Seiberg-Witten differential $d\lambda = kdz$, and its modification $\tilde{\mathcal{F}}$, which is determined rather by the differential $d\tilde{\lambda} = d\lambda + \frac{1}{2}\beta m dz$ with which we have worked in Section IV.(c). We denote by \tilde{a}_i and \tilde{a}_{Di} the quantum order parameters corresponding to $d\tilde{\lambda}$. From (4.34), the various contributions (4.35) to $\tilde{\mathcal{F}}$ are easily identified, up to order $\mathcal{O}(q)$ included. We find

$$\begin{aligned}
\tilde{\mathcal{F}}^{\text{class}} &= \frac{1}{2} \tau \sum_{i=1}^N \tilde{a}_i^2 + \frac{1}{2} m \tau \sum_{i=1}^N \tilde{a}_i \\
\tilde{\mathcal{F}}^{\text{pert}} &= -\frac{1}{8\pi i} \sum_{i,j=1}^N \{(\tilde{a}_i - \tilde{a}_j)^2 \log(\tilde{a}_i - \tilde{a}_j)^2 - (\tilde{a}_i - \tilde{a}_j + m)^2 \log(\tilde{a}_i - \tilde{a}_j + m)^2\} \\
\tilde{\mathcal{F}}^{(1)} &= -\frac{1}{2\pi i} q m^2 \sum_{i=1}^N \prod_{j \neq i} \left(1 - \frac{m^2}{(\tilde{a}_i - \tilde{a}_j)^2}\right)
\end{aligned} \tag{4.36}$$

Now the relation between the order parameters a_i , a_{Di} , and their modified version \tilde{a}_i , \tilde{a}_{Di} is

$$a_i = \tilde{a}_i - \frac{1}{2}m, \quad a_{Di} = \tilde{a}_{Di} - \frac{1}{2}m\tau. \quad (4.37)$$

This implies the following relation between the prepotential \mathcal{F} and its modified version $\tilde{\mathcal{F}}$

$$\mathcal{F}(a) = \tilde{\mathcal{F}}(a + \frac{1}{2}m) - \frac{1}{2}m\tau \sum_{i=1}^N a_i. \quad (4.38)$$

(Although the term $\sum_{i=1}^N a_i$ is physically immaterial (the Wilson effective action depends only on $\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}$), we do not drop it at this point. This is in order to compare eventually the present formulae with those obtained later from a renormalization group equation. Also the $SU(N)$ traceless constraint $\sum_{i=1}^N a_i = 0$ is imposed only *after* differentiating \mathcal{F} in a_i .) Thus we obtain the following expression for \mathcal{F}

$$\begin{aligned} \mathcal{F}^{\text{class}} &= \frac{1}{2}\tau \sum_{i=1}^N a_i^2 + \frac{1}{2}m\tau \sum_{i=1}^N a_i \\ \mathcal{F}^{\text{pert}} &= -\frac{1}{8\pi i} \sum_{i,j=1}^N \{(a_i - a_j)^2 \log(a_i - a_j)^2 - (a_i - a_j + m)^2 \log(a_i - a_j + m)^2\} \\ \mathcal{F}^{(1)} &= -\frac{1}{2\pi i} qm^2 \sum_{i=1}^N \prod_{j \neq i} \left(1 - \frac{m^2}{(a_i - a_j)^2}\right) \end{aligned} \quad (4.39)$$

where we have ignored an additional term $\frac{3}{8}m^2\tau N$, since it is a_i -independent. Thus the prepotential \mathcal{F} retains its form under a shift of $d\lambda$ by a multiple of the form dz , which is another reflection of the invariance of the Lax equation $\dot{L} = [M, L]$ under a shift of L by a multiple of the identity. The contribution $\mathcal{F}^{\text{class}}$ agrees with the well-known form at tree level. The contribution $\mathcal{F}^{\text{pert}}$ is precisely the one expected for a system consisting of gauge multiplet states with masses $|a_i - a_j|$ and of hypermultiplet states with masses $|a_i - a_j + m|$. Notice that both contributions enter with opposite signs as expected. Finally, the contribution $\mathcal{F}^{(1)}$ has not as yet been evaluated starting from conventional field theory methods. It would be interesting to compare our answer with a direct instanton calculation. In Section VI, we present various decoupling limits of the theory and check in particular that $\mathcal{N} = 2$ supersymmetric pure $SU(N)$ gauge theory is recovered upon decoupling the hypermultiplet by letting $m \rightarrow \infty$, and suitably adjusting the coupling τ .

V. THE RENORMALIZATION GROUP EQUATION

In this section, we shall derive a renormalization group equation for the $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory with matter in the adjoint representation. Although the techniques of [11] and [15-18] (see also [19]) were successful in the case of asymptotically free theories with matter in the fundamental representation, they yield in the present case only the homogeneity relation

$$\sum_{i=1}^N a_i \frac{\partial \mathcal{F}}{\partial a_i} + m \frac{\partial \mathcal{F}}{\partial m} - 2\mathcal{F} = 0. \quad (5.1)$$

A much more powerful equation may be produced by considering a renormalization group equation for the variation of \mathcal{F} with respect to τ .

(a) Derivation of the Renormalization Group Equation

The key starting ingredient for our derivation is the following variational formula for the B_i -periods a_{Di} of the Seiberg-Witten differential $d\lambda = \tilde{k}dz$, as defined in (3.14).

$$\begin{aligned} \delta a_{Di} &= \frac{1}{2\pi i} \int \int_{\Gamma} \mu \tilde{k} d\bar{z} \wedge d\omega_i + \sum_{\alpha=1}^N M_{\alpha} V(P_{\alpha}) \phi_i(P_{\alpha}) \\ M_1 &= -(N-1)m \\ M_{\alpha} &= m, \quad \alpha = 2, \dots, N \end{aligned} \quad (5.2)$$

where $\mu = \mu_{\bar{z}}^z d\bar{z} \otimes \frac{d}{dz}$ is the Beltrami differential on Γ induced by a moduli deformation $\tau \rightarrow \tau + \delta\tau$ of the base torus Σ . Here we have represented μ as $\mu = \partial_{\bar{z}} V(z)$, with V a vector field with jump discontinuities across the cycles A_i . We have also introduced the basis $d\omega_i$, $i = 1, \dots, N$, of holomorphic Abelian differentials for the surface Γ which is dual to the cycles A_i , and the associated functions $\phi_i(z)$ by $d\omega_i = \phi_i dz$.

To establish (5.2), we note that the Seiberg-Witten differential in the case of matter in the adjoint representation satisfies the equation

$$\partial_{\bar{z}} d\lambda = -\pi \sum_{i=1}^N M_{\alpha} \delta(z, P_{\alpha}). \quad (5.3)$$

To vary this equation with respect to τ , we recall that the $\partial_{\bar{z}}$ operator on scalars and on one-forms varies respectively by $-\mu \partial_z$ and $-\partial_z \mu$, under a deformation of the complex

structure by a Beltrami differential μ [20,21]. This implies that the local coordinate z is deformed to $z + V(z)$. Thus the variational equation derived from (5.3) is

$$\partial_{\bar{z}}\delta(d\lambda) = -\partial_z(\mu d\lambda) + \pi \sum_{\alpha=1}^N M_\alpha \{V(P_\alpha)\partial_{P_\alpha}\delta(z, P_\alpha) + \bar{V}(P_\alpha)\partial_{\bar{P}_\alpha}\delta(z, P_\alpha)\}. \quad (5.4)$$

Let $E(z, w)$ be the prime form. Since the Green's function for the operator $\partial_{\bar{z}}$ on (1,0) forms is the Szegő kernel $\frac{1}{\pi}\partial_z\log E(z, w)$, we obtain

$$\begin{aligned} \delta(d\lambda)(z) &= \frac{1}{2\pi i} \int_{\Gamma} \int_{\Gamma} \partial_w \partial_z \log E(z, w) (\mu \tilde{k}(w)) d\bar{w} \wedge dw \\ &\quad + \sum_{\alpha=1}^N M_\alpha V(P_\alpha) \partial_{P_\alpha} \partial_z \log E(z, P_\alpha) + \pi \sum_{\alpha=1}^N M_\alpha \bar{V}(P_\alpha) \delta(z, P_\alpha). \end{aligned} \quad (5.5)$$

Now the prime form satisfies the following identities [22]

$$\begin{aligned} \oint_{A_i} dz \partial_z \partial_w \log E(z, w) &= 0 \\ \oint_{B_i} dz \partial_z \partial_w \log E(z, w) &= 2\pi i \phi_i(w). \end{aligned} \quad (5.6)$$

Thus integrating the right hand side of (5.5) over the A_i cycles shows that it has zero A -periods (as it should have), while integrating over the B_i cycles gives the desired equation (5.2), in view of the fact that the punctures P_α do not lie on the B_i cycles.

So far, our considerations have been quite general, and would apply with little change to any deformation of the moduli of Γ . In the present situation, the deformation is of the base torus, and thus the vector field $V(z)$ has the same constant discontinuity across all the A_i cycles. We may rewrite the surface integral in the right hand side of (5.2) as

$$\frac{1}{2\pi i} \int_{\Gamma} \int_{\Gamma} \mu \tilde{k} d\bar{w} \wedge d\omega_i = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \int_{\Gamma \setminus \bigcup_{\alpha=1}^N \{|z - P_\alpha| < \epsilon\}} d(V \tilde{k} d\omega_i).$$

In view of Stokes' theorem, this can be re-expressed as the difference of a line integral around the boundary $\prod_{i=1}^N A_i B_i A_i^{-1} B_i^{-1}$ of the cut surface, and simple contour integrals around the poles P_α of \tilde{k} . Only the poles of \tilde{k} and the discontinuities $V^+ - V^-$ of V across the A_i cycles contribute, since V is continuous across the B_i cycles and the contributions of B_i and B_i^{-1} cancel. Thus

$$\frac{1}{2\pi i} \int_{\Gamma} \int_{\Gamma} \mu \tilde{k} d\bar{w} \wedge d\omega_i = \frac{1}{2\pi i} (V^+ - V^-) \sum_{j=1}^N \oint_{A_j} \tilde{k} d\omega_i - \sum_{\alpha=1}^N M_\alpha V(P_\alpha) \phi_i(P_\alpha)$$

and (5.2) reduces to

$$\delta\left(\frac{1}{2\pi i} \oint_{B_i} d\lambda\right) = \frac{1}{2\pi i} (V^+ - V^-) \sum_{j=1}^N \oint_{A_j} \tilde{k} d\omega_i. \quad (5.7)$$

We can determine the correct value of the jump $V^+ - V^-$ by comparing the preceding formula with the case of moduli deformations of the base torus Σ . In fact, the above argument applies equally well to the variations of the differential dz on the base Σ in place of the variations of $d\lambda$ on the surface Γ . We obtain in this case

$$\delta\left(\oint_B dz\right) = (V^+ - V^-) \oint_A d\omega \quad (5.8)$$

where A and B are now cycles on Σ , and $d\omega$ is the Abelian differential dual to the A cycle. Varying τ with, say, the length $2\omega_1$ of the A cycle fixed, we find

$$V^+ - V^- = -2\omega_1 \delta\tau \quad (5.9)$$

where the - sign is due to the fact that, with the orientation of the A_i and B_i cycles described at the beginning of Section IV, the B cycle actually goes from ω_2 to $-\omega_2$ in the fundamental domain for $\Sigma = \mathbf{C}/2\omega_1\mathbf{Z} + 2\omega_2\mathbf{Z}$. The equation (5.7) becomes

$$\frac{\partial a_{Di}}{\partial \tau} = -\frac{2\omega_1}{2\pi i} \sum_{j=1}^N \oint_{A_j} \tilde{k} d\omega_i \quad (5.10)$$

We can reexpress the equation (5.10) in terms of the prepotential \mathcal{F} . Since the poles of $d\lambda$ are independent of the order parameters a_i , the derivatives of $d\lambda$ with respect to a_i must be holomorphic Abelian differentials. Since the a_i are the A_i periods of $d\lambda$, we have

$$d\omega_i = \frac{1}{2\pi i} \frac{\partial}{\partial a_i} d\lambda = \frac{1}{2\pi i} \frac{\partial \tilde{k}}{\partial a_i} dz. \quad (5.11)$$

In terms of \mathcal{F} , the equation (5.10) is then

$$\frac{\partial}{\partial \tau} \left(\frac{\partial \mathcal{F}}{\partial a_i} \right) = \frac{2\omega_1}{8\pi^2} \frac{\partial}{\partial a_i} \left(\sum_{j=1}^N \oint_{A_j} \tilde{k}^2 dz \right). \quad (5.12)$$

This identifies the renormalization group equation for the $\mathcal{N} = 2$ supersymmetric gauge theory with matter in the adjoint representation to be

$$\frac{\partial \mathcal{F}}{\partial \tau} = \frac{2\omega_1}{8\pi^2} \sum_{j=1}^N \oint_{A_j} \tilde{k}^2 dz. \quad (5.13)$$

We can view the right hand side of (5.13) as an exact formula for the beta function of the theory. It is essentially given by the Hamiltonian of the associated integrable system. Indeed, the summation over all cycles A_j of (5.13) can be replaced by the integral over the single cycle A downstairs of $\tilde{k}_1^2 + \cdots + \tilde{k}_N^2 = \text{Tr} L^2$, where the \tilde{k}_i , $i = 1, \dots, N$, denote the N roots of the defining equation $\det(\tilde{k}I - L(z)) = 0$ of the spectral curve. Explicitly, we have

$$\frac{\partial \mathcal{F}}{\partial \tau} = \frac{2\omega_1}{8\pi^2} \oint_A \text{Tr} L^2 dz = -\frac{\omega_1^2}{2\pi^2} \left(\sum_{i=1}^N p_i^2 - m^2 \sum_{i \neq j} \wp(x_i - x_j) \right). \quad (5.14)$$

where the a_i -independent term $\frac{\omega_1 \eta_1}{2\pi^2} N(N-1)$ has been dropped. Finally, for computational purposes, it is useful to recast the RG equation (5.13) in terms of the variable k in $f(k - \frac{1}{2}\beta m, z) = 0$ rather than the original variable \tilde{k} in $R(\tilde{k}, z) = 0$. Recall that $k = \tilde{k} + mh_1(z) + \frac{1}{2}\beta m$, and thus

$$k^2 = \tilde{k}^2 + 2\tilde{k}(mh_1(z) + \frac{1}{2}\beta m),$$

where we have ignored the term $(mh_1 + \frac{1}{2}\beta m)^2$, since it contributes only an a_i -independent term to the prepotential. The integral over the sum of all A_j cycles of the second term on the right hand side of the above equation can be again replaced by an integral over the cycle A downstairs, with \tilde{k} replaced by the trace $\sum_{i=1}^N p_i$ of the matrix $L(z)$ (c.f. (2.11)). We obtain

$$\sum_{j=1}^N \oint_{A_j} 2\tilde{k}(mh_1(z) + \frac{1}{2}\beta m) dz = 2 \left(\sum_{j=1}^N p_j \right) \oint_A (mh_1(z) + \frac{1}{2}\beta m) dz = 0,$$

in view of the transformation law (3.3) for the ϑ_1 -function, and the fact that $h_1(z) = \partial_z \log \vartheta_1(\frac{z}{2\omega_1} | \tau)$. Thus we may use either \tilde{k}^2 or k^2 in the expression (5.13) for the beta function.

As an immediate check, we determine the classical part $\mathcal{F}^{\text{class}}$ of the prepotential (4.42) from the renormalization group equation, leaving the more interesting derivation of the one-instanton and two-instanton contributions to the next section. Ignoring then $\mathcal{O}(q)$ terms, we find the following value for the beta function, in view of the expansion (4.13) and (4.19) for dz

$$\frac{\partial \mathcal{F}}{\partial \tau} = \frac{1}{2\beta^2} \sum_{i=1}^N k_i^2 + \mathcal{O}(q).$$

Since (4.21) implies that $\beta^{-1}k_i = a_i + \frac{1}{2}m + \mathcal{O}(q)$ (recall that the left hand side of (4.21) is actually \tilde{a}_i , and that a_i and \tilde{a}_i are related by (4.37)), we obtain

$$\frac{\partial \mathcal{F}}{\partial \tau} = \frac{1}{2} \sum_{i=1}^N a_i^2 + \frac{1}{2}m \sum_{i=1}^N a_i + \mathcal{O}(q),$$

in agreement with (4.39).

Henceforth, it is convenient to set $\beta = 1$, i.e. $\omega_1 = -i\pi$, since the prepotential is independent of β anyway. Our final renormalization group equation takes the form

$$\frac{\partial \mathcal{F}}{\partial \tau} = \frac{1}{4\pi i} \sum_{j=1}^N \oint_{A_j} \tilde{k}^2 dz = \frac{1}{4\pi i} \sum_{j=1}^N \oint_{A_j} k^2 dz. \quad (5.15)$$

The power of this formula lies in the fact that all terms in an expansion in powers of q may be evaluated using residue methods only, just as was the case for the calculation of the A_j periods. To show how this works, we produce now a calculation of the 1- and 2-instanton contributions.

(b) 1- and 2- Instanton Results from the Renormalization Group Equation

In this section, we evaluate the 1- and 2-instanton contributions to the prepotential, using the renormalization group equation of (5.15). As stated there, we set $\beta = 1$. The only objects we need to calculate are the a_i quantum order parameters, as integrals over A_i cycles of the differential $k dz$, and the RG beta function in (5.15) as integrals over A_i cycles of the differential $k^2 dz$. Both calculations are carried out by residue methods only ! As in Section IV, we choose k as independent variable, and need to express $dz = d \log w$ as a function of k . This is done with the help of (4.13), and the expansion of $\log y$ is obtained from (4.14) and (4.15). Since the A_i cycles may be chosen away from the poles k_i by a distance of order $\mathcal{O}(q^0)$, it suffices to use an expansion for $\log w$ or $\log y$ in powers of q , and for $k - k_i$ or order $\mathcal{O}(q^0)$. In particular, we do not need to worry about k coming close to k_i by a distance of order $\mathcal{O}(q^{\frac{1}{2}})$, as we had to in equations (4.17) to (4.19). Taking these considerations into account, we find

$$\log y = q(-\eta_1 + \bar{\eta}_1) + \frac{1}{2}q^2(-3\eta_1^2 + 3\bar{\eta}_1^2) + \mathcal{O}(q^3) \quad (5.16)$$

where

$$\eta_1 = \frac{H(k+m)H(k-m)}{H(k)^2} \quad \bar{\eta}_1 = \frac{H(k)H(k-2m)}{H(k-m)^2} \quad (5.17)$$

The differential dz thus becomes

$$dz = d \log H(k) - d \log H(k-m) + q(-\eta_1' + \bar{\eta}_1') dk + q^2(-3\eta_1 \eta_1' + 3\bar{\eta}_1 \bar{\eta}_1') dk. \quad (5.18)$$

where the prime stands for derivation with respect to k . Now, in evaluating the A-periods using residue methods at the zeros of $H(k)$, the contribution $d \log H(k-m)$ and terms

involving $\bar{\eta}_1$ never enter since they do not exhibit poles at $k = k_i$. Thus, we are left with dependence only on $H(k)$ and on the function η_1 . Residue calculations at the poles k_i will in fact only involve the functions

$$\bar{S}_i(k) \equiv \frac{H(k+m)H(k-m)}{H_i(k)^2}. \quad (5.19)$$

(Notice that with this notation, we have $(k_i^{(1)})^2 = 2\bar{S}_i(k_i)$ and $k_i^{(2)} = 2\bar{S}_i'(k_i)$, according to (4.11).) We find, again to order $\mathcal{O}(q^2)$:

$$\begin{aligned} a_i &= k_i + q\bar{S}_i'(k_i) + \frac{1}{4}q^2\{\bar{S}_i^2\}'''(k_i) \\ \frac{\partial \mathcal{F}}{\partial \tau} &= \sum_{i=1}^N \left\{ \frac{1}{2}k_i^2 + qk_i\bar{S}_i'(k_i) + q\bar{S}_i(k_i) + \frac{1}{4}q^2k_i\{\bar{S}_i^2\}'''(k_i) + \frac{3}{4}q^2\{\bar{S}_i^2\}''(k_i) \right\} \end{aligned} \quad (5.20)$$

where the prime denotes taking a derivative with respect to k and setting $k = k_i$ afterwards. It now simply remains to recast the second equation in (5.20) in terms of variable a_i only, and this is achieved using the first equation in (5.20). It is convenient to define the function $S_i(a)$, which is the analogue of $\bar{S}_i(k)$, but with all variables k_i replaced by a_i ; explicitly, we have

$$S_i(a) = \frac{\prod_{j=1}^N (a - a_j + m)(a - a_j - m)}{\prod_{j \neq i}^N (a - a_j)^2} \quad (5.21)$$

In terms of this function, we find

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial \tau} &= \sum_{i=1}^N \left\{ \frac{1}{2}a_i^2 + qS_i(a_i) + q^2S_i'(a_i)^2 + \frac{3}{2}q^2S_i(a_i)S_i''(a_i) \right\} \\ &\quad - q^2 \sum_{i,j=1}^N S_j'(a_j) \frac{\partial}{\partial a_j} S_i(a_i) \end{aligned} \quad (5.22)$$

which may be integrated with respect to τ (keeping a_i fixed) in a straightforward way. We find

$$\begin{aligned} \mathcal{F} &= \mathcal{F}^{\text{class}} + \mathcal{F}^{\text{pert}} + \frac{1}{2\pi i} \sum_{i=1}^N \left\{ qS_i(a_i) + \frac{1}{2}q^2S_i'(a_i)^2 + \frac{3}{4}q^2S_i(a_i)S_i''(a_i) \right\} \\ &\quad - \frac{1}{4\pi i} q^2 \sum_{i,j=1}^N S_j'(a_j) \frac{\partial}{\partial a_j} S_i(a_i) \end{aligned} \quad (5.22)$$

It will be convenient for our study of various decouplings in Section VI, to recast (5.22) by rearranging the 2-instanton contributions in the following form.

$$\begin{aligned} \mathcal{F} = \mathcal{F}^{\text{class}} + \mathcal{F}^{\text{pert}} + \frac{1}{2\pi i} \sum_{i=1}^N \left\{ q S_i(a_i) + \frac{1}{4} q^2 S_i(a_i) \frac{\partial^2 S_i(a_i)}{\partial a_i^2} \right\} \\ + \frac{1}{2\pi i} q^2 \sum_{i \neq j=1}^N S_i(a_i) S_j(a_j) \left[\frac{1}{(a_i - a_j)^2} - \frac{1}{2} \frac{1}{a_i - a_j + m} - \frac{1}{2} \frac{1}{a_i - a_j - m} \right]. \end{aligned} \quad (5.23)$$

It is easy to see that the classical part is that of (4.39), and that the 1-instanton contribution (linear in q) also agrees with (4.39). The perturbative part $\mathcal{F}^{\text{pert}}$ is independent of τ and cannot be obtained from the renormalization group equation; it was taken over from (4.39). The 2-instanton contribution (quadratic in q) is new. In Section VI, we shall show that upon decoupling the hypermultiplet by sending $m \rightarrow \infty$, (5.23) converges to the prepotential for the pure $\mathcal{N} = 2$ theory with gauge group $SU(N)$, as obtained in [7,11].

VI. DECOUPLING LIMITS AND PRODUCT GAUGE GROUPS

The independent parameters in the $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory are the complex gauge coupling τ , the hypermultiplet mass parameter m , and the quantum order parameters a_i , $i = 1, \dots, N$ (or equivalently the classical order parameters k_i). The masses of the gauge multiplet and hypermultiplet states are respectively given by $|a_i - a_j|$ and $|a_i - a_j + m|$. By taking various singular limits of the parameters, subsets of the states of the theory may be given infinitely large mass and made to decouple. What remains after decoupling is a different $\mathcal{N} = 2$ supersymmetric gauge theory. To be more precise, singular limits of the parameters may induce the following two effects.

- (1) States whose mass tends to ∞ disappear from the spectrum and decouple from the dynamics of the theory. The gauge group of the remaining theory is in general a subgroup (which need not be a simple group) of the original gauge group $SU(N)$.
- (2) The dynamics associated to one or several of the gauge subgroups may freeze out when the effective coupling of this gauge subgroup tends to zero. The gauge states of the corresponding gauge subgroup become non-interacting, and the adjoint scalar fields that belong to the $\mathcal{N} = 2$ multiplet of the gauge subgroup are frozen to their vacuum expectation values.

In this section, we shall describe various decoupling limits of the $\mathcal{N} = 2$ theory with a massive adjoint hypermultiplet. We begin by disposing of some cases which are uninteresting. First, keeping m and a_i finite and letting $\tau \rightarrow \infty$ produces a free theory. Second,

keeping τ and m finite, and letting some $a_i \rightarrow \infty$, produces an $\mathcal{N} = 2$ gauge theory with an adjoint hypermultiplet and a gauge group which is a product $SU(N_1) \times \cdots \times SU(N_p)$ subgroup (and possibly $U(1)$ factors) of $SU(N)$, with decoupled dynamics between different factors. Third, keeping m finite, and letting $\tau \rightarrow \infty$ and some of the $a_i \rightarrow \infty$, we recover again a free theory.

Thus, to obtain interesting decoupling limits, we must let $\tau \rightarrow \infty$ and at the same time $m \rightarrow \infty$ in a related way. Notice that in this case either the gauge mass $|a_i - a_j|$ or the hypermultiplet mass $|a_i - a_j + m|$ must tend to ∞ , since both could not be kept finite at the same time. Thus for given ij , either the gauge or the hypermultiplet state must decouple. We distinguish two cases :

- (a) All a_i remain finite, in which case the gauge group remains $SU(N)$, but the full hypermultiplet decouples. We show in subsection (a) below that the pure $SU(N)$ theory is indeed recovered. This decoupling limit was also exhibited in [2] using the curve obtained through the Hitchin system.
- (b) Some a_i are also sent to ∞ in such a way that certain hypermultiplet masses remain finite. In this limit, the gauge group $SU(N)$ is broken to a subgroup of the type $SU(N_1) \times \cdots \times SU(N_p)$. (We shall see that $U(1)$ factors, which in principle could appear, actually always decouple.) The remaining hypermultiplets may be in fundamental representations of one of the gauge group factors or in bi-fundamental representations of two of the gauge factors (not all such bi-fundamentals are allowed though). In subsection (b) below, we analyze this case when two factors arise (this includes the case where the dynamics of one of the factors freezes out as explained in (2) above.) In subsection (c) we analyze the general case, and discuss as an application the case of 3 factors in (d).

We shall now examine each of these cases in turn.

(a) Decoupling the full hypermultiplet

The pure $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory is recovered by decoupling the full hypermultiplet in the limit where $\tau \rightarrow \infty$, $m \rightarrow \infty$ while keeping constant the parameters a_i and Λ :

$$\Lambda^{2N} = (-)^N m^{2N} q \qquad q = e^{2\pi i \tau}. \quad (6.1)$$

Notice that since $q \rightarrow 0$ in this limit, it is equivalent to keep the classical order parameters k_i fixed. The curve of (4.5) then converges to the curve of the pure theory upon scaling

the variable w in such a way that t , as defined below, is kept fixed

$$H(k) - t - \frac{1}{t}\Lambda^{2N} = 0 \quad w = t(-m)^{-N}. \quad (6.2)$$

The Seiberg-Witten differential directly follows from the same change of variable $z = \log w$ to t in (6.2) : $d\lambda = kd \log t$.

The limit of the effective prepotential in (5.23) is readily obtained by first establishing that

$$qS_i(a_i) \rightarrow \Lambda^{2N} \prod_{i \neq j}^N \frac{1}{(a_i - a_j)^2}. \quad (6.3)$$

and is found to agree (up to an irrelevant additive constant) with expression (4.34a,b,c) of [7] for the pure case $N_f = 0$.

Next, we show that the renormalization group equation of (5.15) reduces to the renormalization group equation for the case of the pure theory. As the curve for the adjoint case converges to that of the pure case (6.2), the curve becomes hyperelliptic. The sum over the A_i -cycles in (5.15) may be deformed into a single contour encircling all A_i branch cuts, which in turn may be deformed into a contour around $k = \infty$. The sum over the corresponding contour integrals may now be evaluated by residue methods around $k = \infty$. Upon defining s_2 by $H(k) = k^N + s_2 k^{N-2} + \mathcal{O}(k^{N-3})$, and using the relation (6.1) to convert the variation with respect to τ into a variation with respect to Λ while keeping m fixed, we find

$$\frac{\partial \mathcal{F}}{\partial \log \Lambda} = -\frac{2N}{2\pi i} s_2 \quad (6.4)$$

in agreement with equation (3.20) of [11].

Finally, we note that the decoupling of the full hypermultiplet may also be carried out directly on the Lax operator $L(z)$ in (2.11), by taking the well-known singular limit of the Calogero-Moser system to the affine Toda system [23] for group $SU(N)$. In this way, we recover the curve for the pure $SU(N)$ theory, obtained as the spectral curve of the Lax operator for affine Toda for $SU(N)$ in [24].

(b) Decoupling $SU(N) \rightarrow SU(N_1) \times SU(N_2)$

Next, we consider a singular limit, in which the quantum order parameters a_i (or equivalently the classical order parameters k_i) fall into two groups with $i = 1, \dots, N_1$ and $j = N_1 + 1, \dots, N = N_1 + N_2$, (with $N_1, N_2 \geq 1$). The groups are such that as $m \rightarrow \infty$, we have $(a_i - a_j)/m \rightarrow 0$ when i, j belong to the same group and $(a_i - a_j)/m \rightarrow \pm 1$ when i, j belong to different groups. It is clear that the light gauge states fill out a multiplet of the

gauge group $SU(N_1) \times SU(N_2) \times U(1)$, and that the light hypermultiplet states transform under the bi-fundamental representation $(\mathbf{N}_1, \overline{\mathbf{N}}_2) \oplus (\overline{\mathbf{N}}_1, \mathbf{N}_2)$ of the semi-simple part of the gauge group.

The curve and effective prepotential may be obtained as a limit of the case with adjoint hypermultiplet. However, special care is required in the analysis of the three gauge couplings of the gauge group $SU(N_1) \times SU(N_2) \times U(1)$. Indeed, one or several of the couplings may flow to zero in the limit and the dynamics of the corresponding gauge group may freeze out, as discussed in (2) above. In fact, this will always be the case for the $U(1)$ part of the gauge group, but may also occur for one of the simple factors.

To analyze the singular limits quantitatively, we decompose the classical (equivalently quantum) order parameters in terms of order parameters x_i and y_j of the gauge groups $SU(N_1)$ and $SU(N_2)$ respectively

$$\begin{aligned} k_i &= v_1 + x_i & i &= 1, \dots, N_1 \\ k_{N_1+j} &= v_2 + y_j & j &= 1, \dots, N_2. \end{aligned} \tag{6.5}$$

Uniqueness of the decomposition is achieved by imposing the conditions $\sum_i x_i = \sum_j y_j = 0$, so that $N_1 v_1 + N_2 v_2 = 0$. The order parameter $v \equiv v_1 - v_2$ is associated with the $U(1)$ factor of the gauge group.

The Limiting Curve

We now assume that x_i , y_j and $\mu = v - m$ are kept fixed in the limit, and determine the behavior of τ and m (which tend to ∞) that can yield interesting limits. First, we determine the limiting curves and then analyze which gauge couplings tend to zero. The limit of the curve (4.5) for adjoint hypermultiplet is obtained by deriving the large m behavior of the polynomial $H(k)$. It is convenient to work with the variable $x = k - v_1$, since it has a finite limit as $m \rightarrow \infty$.

$$H(k) = \prod_{i=1}^{N_1} (x - x_i) \prod_{j=1}^{N_2} (x + m + \mu - y_j), \quad k = x + v_1. \tag{6.6}$$

We define the polynomials

$$\begin{aligned} A(x) &\equiv \prod_{i=1}^{N_1} (x - x_i) \\ B(x) &\equiv \prod_{j=1}^{N_2} (x + \mu - y_j), \end{aligned} \tag{6.7}$$

and obtain the leading large m behavior of $H(k)$ as a function of $A(x)$ and $B(x)$:

$$\begin{aligned} H(k) &\sim m^{N_2} A(x) \\ H(k-m) &\sim (-m)^{N_1} B(x) \\ H(k-nm) &\sim c_n m^{N_1+N_2}, \quad c_n = (-n)^{N_1} (-n+1)^{N_2} \quad n \neq 0, 1. \end{aligned} \tag{6.8}$$

It is convenient to redefine w in terms of a parameter t that has a finite limit as $m \rightarrow \infty$:

$$w \equiv tm^{N_2-N_1}. \tag{6.9}$$

The leading large m behavior of the curve is now given by

$$A(x) - t(-)^{N_1} B(x) + \sum_{n \in \mathbf{Z}, n \neq 0, 1} (-)^n q^{\frac{1}{2}n(n-1)} t^n c_n m^{\nu_n} = 0 \tag{6.10}$$

where $\nu_n = N_1 + n(N_2 - N_1)$.

It remains to derive the behavior of q as a function of m in the limit. To do so, we assume, without loss of generality, that $N_2 \leq N_1$. In (6.10), the orders $n = -l$ and $n = l+1$ with $l \geq 0$ enter with the same q -dependence, but, we have $\nu_{-l} - \nu_{l+1} = -(2l+1)(N_2 - N_1) \geq 0$. Thus, to order q , the term with $n = -1$ dominates in (6.10) over the term with $n = 2$ if $N_2 < N_1$, while both terms are on the same order for $N_2 = N_1$. In either case, an interesting limit is obtained by holding Λ fixed in the following q dependence

$$qm^{2N_1-N_2} \equiv \Lambda^{2N_1-N_2}. \tag{6.11}$$

It is easy to see that with this dependence, all terms in (6.10) with $n \neq -1, 0, 1, 2$ tend to 0. The limiting curve is

$$A(x) - t(-)^{N_1} B(x) - 2^{N_2} \Lambda^{2N_1-N_2} \left[\frac{1}{t} - t^2 (-)^N \left(\frac{m^3}{2} \right)^{N_2-N_1} \right] = 0 \tag{6.12}$$

Of course, when $N_2 < N_1$, the last term in (6.12) may be dropped.

Flow of Couplings, and Freezing out of the $U(1)$

To analyze the physical system governed by curve (6.12), we study the behavior of the three couplings, associated with the gauge groups $SU(N_1)$, $SU(N_2)$ and $U(1)$ respectively. As the coupling q tends to zero in the limit of $m \rightarrow \infty$, as given by (6.11), it suffices to use perturbation theory to do so, retaining only $\mathcal{F}^{\text{class}} + \mathcal{F}^{\text{pert}} \equiv i\Phi/8\pi$ in the expansion of the prepotential (4.36). To this order, the quantum order parameters a_i in (4.36) may be replaced by the classical order parameters k_i . In order to keep track of dimensionful

parameters in a consistent way, it is convenient to introduce an *arbitrary* scale parameter inside the logarithmic terms, which we shall choose to be $\Lambda > 0$, introduced in (6.11). The function Φ splits up into a part with a manifestly finite limit

$$\begin{aligned} \Phi^{\text{lim}} = & \sum_{i,j=1}^{N_1} (x_i - x_j)^2 \log(x_i - x_j)^2 / \Lambda^2 + \sum_{i,j=1}^{N_2} (y_i - y_j)^2 \log(y_i - y_j)^2 / \Lambda^2 \\ & - \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (x_i - y_j + \mu)^2 \log(x_i - y_j + \mu)^2 / \Lambda^2, \end{aligned} \quad (6.13)$$

a part which involves only quadratic dependence on x_i , y_j and μ

$$\Phi^{\text{quad}} = C_x \sum_{i=1}^{N_1} x_i^2 + C_y \sum_{j=1}^{N_2} y_j^2 + C_\mu \mu^2, \quad (6.14)$$

and a part which is independent of x_i , y_j and μ and which may thus be dropped. The coefficients C are easily computed and we find

$$\begin{aligned} C_x &= -4\pi i\tau + (N_2 - 2N_1)(\log m^2 / \Lambda^2 + 3) - N_2 \log 4 \\ C_y &= -4\pi i\tau + (N_1 - 2N_2)(\log m^2 / \Lambda^2 + 3) - N_1 \log 4 \\ C_\mu &= \{-4\pi i\tau / N + \log m^2 / 4\Lambda^2 + 3\} N_1 N_2. \end{aligned} \quad (6.15)$$

It is clear from (6.15) that as $m \rightarrow \infty$ and $\text{Im}\tau \rightarrow \infty$, we will always have $C_\mu \rightarrow \infty$. This means that the $U(1)$ gauge coupling tends to zero so that the dynamics of this part of the gauge group freezes out. The field μ (which was the scalar component of the $\mathcal{N} = 2$ vector multiplet associated with the $U(1)$ factor) freezes to a constant. Thus, the gauge group of the limiting theory is only $SU(N_1) \times SU(N_2)$.

The Case $N_1 = N_2$

Next, we investigate under which conditions the effective couplings for both $SU(N_1)$ and $SU(N_2)$ remain finite as $m, \text{Im}\tau \rightarrow \infty$. This will require that both C_x and C_y converge to a finite limit. It is clear from (6.15) that *this can happen if and only if $N_1 = N_2$* . In that case, both couplings are the same, and the theory has $SU(N_1) \times SU(N_1)$ gauge group with a hypermultiplet in the bi-fundamental representation $(\mathbf{N}_1, \bar{\mathbf{N}}_1) \oplus (\bar{\mathbf{N}}_1, \mathbf{N}_1)$, with mass μ .

The curve in this case is given by (6.12), and simplifies to

$$A(x) - t(-)^{N_1} B(x) - 2^{N_1} \Lambda^{N_1} \left[\frac{1}{t} - t^2 \right] = 0 \quad (6.16)$$

Its form is easily seen to coincide with the model for the product group $SU(N_1) \times SU(N_1)$ solved by Witten using M-theory and D-brane technology [12]. The same form was also

derived by Katz, Mayr, and Vafa [25] using compactifications of Type II strings on Calabi-Yau manifolds and the mirror symmetry of K3.

Although the curve (6.16) is relatively complicated (say, compared to the curve (3.1) of the $SU(N)$ theory with matter in the fundamental representation), its effective prepotential to any order of instanton corrections can be easily read off from the corresponding prepotential for the $SU(N)$ theory with matter in the adjoint representation by taking the limit $m \rightarrow \infty$, $q \rightarrow 0$, $qm^{N_1} = \Lambda^{N_1}$ fixed. It is convenient to introduce the notation $x_i = x_i^1$, $y_i = x_i^2$. For each I , the index i in x_i^I should be viewed as running over a range $1 \leq i \leq N_I$, which we denote more informally by “ $i \in I$ ”. We set

$$\begin{aligned} A_i^I(x) &= \prod_{\substack{j \neq i \\ j \in I}} (x - x_j^I) & B^I(x) &= \prod_{\substack{j \in J \\ |I-J|=1}} (\mu \pm (x - x_j^J)) \\ S_i^I(x) &= \frac{B^I(x)}{A_i^I(x)^2}, \end{aligned} \tag{6.17}$$

where the sign \pm in the expression $B^I(x)$ is the same as the sign of $J - I$. Then the limit of the function $S_i(a)$ in (5.21) is given by

$$qS_i(a_i) \rightarrow (-2\Lambda)^{N_1} S_i^I(x_i^I).$$

We may now substitute in the expansion (5.23) and obtain this way the prepotential to two-instanton order for the $SU(N) \times SU(N)$ theory with a hypermultiplet of mass μ in the bi-fundamental representation $(\mathbf{N}_1, \bar{\mathbf{N}}_1) \oplus (\bar{\mathbf{N}}_1, \mathbf{N}_1)$

$$\begin{aligned} \mathcal{F} &= \mathcal{F}^{\text{class}} + \mathcal{F}^{\text{pert}} \\ &+ \frac{1}{2\pi i} \sum_{I=1,2} \left[\sum_{i \in I} (-2\Lambda)^{N_1} S_i^I(x_i^I) + \frac{1}{4} (2\Lambda)^{2N_1} S_i^I(x_i^I) \left(\frac{\partial}{\partial x_i^I} \right)^2 S_i^I(x_i^I) \right. \\ &\quad \left. + (2\Lambda)^{2N_1} \sum_{\substack{i \neq j \\ i, j \in I}} \frac{S_i^I(x_i^I) S_j^I(x_j^I)}{(x_i^I - x_j^I)^2} \right]. \end{aligned}$$

The Case $N_2 < N_1$

Substituting the limiting behavior of q in (6.11), we find for the coefficients

$$\begin{aligned} C_x &= 3(N_2 - 2N_1) + \pi i N_1 - N_2 \log 4 \\ C_y &= 3(N_1 - 2N_2) + \pi i N_1 - N_1 \log 4 + 3(N_1 - N_2) \log \frac{m^2}{\Lambda^2} \end{aligned}$$

Clearly, C_y diverges as $m \rightarrow \infty$, the coupling constant for the gauge group $SU(N_2)$ freezes out to zero and the fields y_j , $j = 1, \dots, N_2$ are frozen to their constant expectation values.

C_x on the other hand has a finite limit. Thus the remaining gauge group is the *color group* $SU(N_1)$, while $SU(N_2)$ effectively acts as a (approximate) *flavor group*. Denoting $\mu - y_j \equiv m_j$, we readily recognize the curve (6.12) for this case

$$A(x) - tB(x) - 2^{N_2} \Lambda^{2N_1 - N_2} \frac{1}{t} = 0 \quad (6.18)$$

as the curve for an $\mathcal{N} = 2$ supersymmetric theory with color group $SU(N_c)$, $N_c = N_1$, and $N_f = N_2$ flavors in the fundamental representation of $SU(N_c)$. Notice that the models we have obtained this way are restricted by $N_f < N_c$!

As in the previous case, the effective prepotential for this model is also easily recovered from the general expression for adjoint hypermultiplets in (5.23) up to 2 instanton order by taking the limit $m \rightarrow \infty$, $\tau \rightarrow \infty$ with Λ fixed and substituting into the general form (5.23). We obtain formulas similar to (6.17), with the expression $\mu - x_j^2 = m_j$ being interpreted this time however as the mass m_j of the hypermultiplet. The result clearly agrees with the one obtained for this case in [7] for $N_c = N_1$ and $N_f = N_2$, subject to the condition $N_f < N_c$, which is inherent to the construction here.

(c) Decoupling $SU(N) \rightarrow SU(N_1) \times \cdots \times SU(N_p)$

We now treat the general case of decoupling, in which τ , m and k_i (or equivalently a_i) are all allowed to tend to infinity. Unless there are special relations between the order parameters k_i , such limits will either be free field theories or will be models containing several mutually non-interacting systems. All such systems may be decomposed into basic irreducible systems in which the k_i form a *linear chain*.

To define a linear chain, we divide all the order parameters k_i for $i = 1, \dots, N = N_1 + \cdots + N_p$ into p groups with $i = i_1 = 1, \dots, N_1$ in the first group, $i = N_1 + i_2$ with $i_2 = 1, \dots, N_2$ in the second group and more generally $i = N_1 + N_2 + \cdots + N_{I-1} + i_I$ with $i_I = 1, \dots, N_I$ in the group indexed by $I = 1, \dots, p$. (We shall assume that $N_I \geq 1$ for all $I = 1, \dots, p$ and use the notation $N_0 = N_{p+1} = 0$ for convenience.) In analogy with (6.5), we decompose the k_i as follows

$$k_{\{N_1 + \cdots + N_{I-1} + i_I\}} = v_I + x_{i_I}^I \quad \begin{matrix} I = 1, \dots, p \\ i_I = 1, \dots, N_I \end{matrix} \quad (6.19)$$

and fix the decomposition uniquely by requiring in addition that $\sum_{i_I=1}^{N_I} x_{i_I}^I = 0$, so that $\sum_I N_I v_I = 0$. A linear chain is such that as $m \rightarrow \infty$, the x_{i_I} are fixed and the v_I are linked by the relations

$$v_I - v_{I+1} = m + \mu_I \quad I = 1, \dots, p-1 \quad (6.20)$$

where μ_I is also kept fixed. Notice that the definition of a linear chain gives an ordering to the N_I .

Flow of Couplings, and Freezing out of the $U(1)$'s

We shall now analyze the decoupling limit for an arbitrary linear chain, and determine the behavior of q as $m \rightarrow \infty$, as well as the conditions on the integers N_I for this theory to be irreducible. We shall begin by obtaining the prepotential to classical and perturbative order for this arrangement of the order parameters. We define $\mathcal{F}^{\text{class}} + \mathcal{F}^{\text{pert}} \equiv i\Phi/8\pi$, and split up Φ into a part with a manifestly finite limit

$$\begin{aligned} \Phi^{\text{lim}} = & \sum_{I=1}^p \sum_{i,j=1}^{N_I} (x_i^I - x_j^I)^2 \log(x_i^I - x_j^I)^2 / \Lambda^2 \\ & - \sum_{I=1}^p \sum_{i=1}^{N_I} \sum_{j=1}^{N_{I+1}} (x_i^I - x_j^{I+1} + \mu_I)^2 \log(x_i^I - x_j^{I+1} + \mu_I)^2 / \Lambda^2, \end{aligned} \quad (6.21)$$

a part which involves only quadratic dependence on x_i^I and only quadratic and linear dependence on μ_I

$$\Phi^{\text{quad}} = C_{\mu_I}^0 + \sum_{I=1}^{p-1} C_{\mu_I} \mu_I^2 - 4\pi i\tau \sum_{I=1}^{p-1} N_{I+1} (\mu_1 + \cdots + \mu_I)^2 + \sum_{I=1}^p C_{x_I} \sum_{i=1}^{N_I} (x_i^I)^2, \quad (6.22)$$

and a part which is independent of $x_{i_I}^I$ and μ_I , and which may be omitted. The coefficients C_{x_I} and C_{μ_I} are given by

$$\begin{aligned} C_{x_I} = & C_{x_I}^0 - 4\pi i\tau + (N_{I+1} - 2N_I + N_{I-1}) \log \frac{m^2}{\Lambda^2} \\ C_{\mu_I} = & N_I N_{I+1} \log \frac{m^2}{\Lambda^2} \end{aligned} \quad (6.23)$$

The remaining coefficient $C_{x_I}^0$ depends only on the N_I while the coefficient $C_{\mu_I}^0$ depends upon N_I as well as upon the parameters μ_I , but both are independent of m and τ . Neither coefficients will be needed, and we shall not give them here.

Now, it is immediately apparent that as soon as $\tau \rightarrow \infty$ and $m \rightarrow \infty$, we will have that all the coefficients $C_{\mu_I} \rightarrow \infty$ or more precisely $\text{Re}(C_{\mu_I}) \rightarrow +\infty$. At the same time, the real part of all coefficients in the third term in (6.22) also tend to $+\infty$. This means that all the couplings associated with the $U(1)$ factors that would classically arise due to the breaking in (6.19) in fact freeze out from the theory, and the true gauge group is reduced to $SU(N_1) \times \cdots \times SU(N_p)$. The fields μ_I are frozen to their constant expectation values.

Next, unless the coefficients $N_{I+1} - 2N_I + N_{I-1}$ are all the same for $I = 2, \dots, p-1$, it will not be possible to choose a behavior for the coupling τ such that the p simple components $SU(N_I)$ remain mutually interacting. In that case, the linear chain will break into two mutually non-interacting smaller linear chains. We shall further justify this assertion below when deriving the limiting curve. (The coefficients for $I = 1$ and $I = p$ are not required to satisfy this condition since they correspond to the groups at the end of the linear chain, and their freezing out will not break the linear chain.) Thus, we shall assume that there exists an integer K such that

$$N_{I+1} - 2N_I + N_{I-1} = -K, \quad I = 2, \dots, p-1 \quad (6.24)$$

Clearly, for the couplings to have a finite limit, we need to have $K > 0$, otherwise, each of the couplings associated with the groups $SU(N_I)$ will tend to zero as $\tau \rightarrow \infty$, and we would end up with a free theory. Relation (6.24) requires a rather peculiar quadratic dependence of the N_I on I , given for $I = 2, \dots, p-1$ by

$$N_I = \frac{1}{2}K(I-1)(p-I) + \frac{N_p - N_1}{p-1}(I-1) + N_1, \quad (6.25)$$

with two parameters N_1 and N_p , in addition to the number p . Notice that (6.25) imposes constraints between these parameters, arising from divisibility conditions : $2(N_p - N_1)$ must be divisible by $p-1$, and the quotient must be even (odd) when Kp is even (odd).

The Limiting Curves

Deriving the limiting curves for the $SU(N_1) \times \dots \times SU(N_p)$ theories may be done in complete parallel to the case with just two factors, carried out above. The polynomial $H(k)$, in terms of which the curve (4.5) is formulated, is calculated in the approximation of large m first. We have

$$H(x + v_1) = \prod_{I=1}^p \prod_{i=1}^{N_I} (x + (I-1)m + M_I - x_i^I) \quad (6.26)$$

where we use the notation $M_I \equiv \mu_1 + \dots + \mu_{I-1}$, with $M_0 = 0$. The $m \rightarrow \infty$ limit is conveniently formulated in terms of the following p polynomials

$$A_I(x) \equiv \prod_{i=1}^{N_I} (x + M_I - x_i^I) \quad I = 1, \dots, p, \quad (6.27)$$

and the following constants c_n , defined for given N_I :

$$\begin{aligned} c_n &\equiv \prod_{I \neq 1+n}^p (I-1-n)^{N_I} & n = 0, 1, \dots, p-1 \\ c_n &\equiv \prod_{I=1}^p (I-1-n)^{N_I} & n \neq 0, 1, \dots, p-1. \end{aligned} \quad (6.28)$$

The limits of $H(x + v_1 - nm)$ are then given by

$$\begin{aligned} H(x + v_1 - nm) &= c_n m^{N-N_{n+1}} A_{n+1}(x) & n = 0, 1, \dots, p-1 \\ H(x + v_1 - nm) &= c_n m^N & n \neq 0, 1, \dots, p-1 \end{aligned} \quad (6.29)$$

Just as for the case $p = 2$, it is also necessary to redefine w according to

$$w \equiv tm^{N_2-N_1}. \quad (6.30)$$

Substituting this limiting behavior into the formula for the curve (4.5), we obtain

$$\begin{aligned} 0 &= \sum_{n=0}^{p-1} q^{\frac{1}{2}n(n-1)} (-)^n t^n m^{N_1-N_{n+1}+n(N_2-N_1)} c_n A_{n+1}(x) \\ &+ \sum_{n \neq 0, 1, \dots, p-1}^{\infty} q^{\frac{1}{2}n(n-1)} (-)^n t^n m^{N_1+n(N_2-N_1)} c_n \end{aligned} \quad (6.31)$$

It is now clear, directly from equation (6.31) for the limiting curve, how the quadratic dependence of N_I on I arises. In order to have a limit where all gauge groups remain mutually interacting, we must retain the dependence of the curve on all the polynomials $A_I(x)$ for all $I = 1, \dots, p$. This can be achieved only if the combination Λ , defined by

$$q^{\frac{1}{2}n(n-1)} = \left(\frac{\Lambda}{m}\right)^{N_1-N_{n+1}+n(N_2-N_1)}, \quad n = 0, 1, \dots, p-1, \quad (6.32)$$

remains finite and can be held fixed. The equation is trivially satisfied for $n = 0, 1$, and for $n = 2, \dots, p-1$ reduces to (6.24) for unspecified value of K . Henceforth, we shall assume that (6.24) is satisfied for some $K > 0$. The equation for the limiting curve then becomes

$$0 = \sum_{n=0}^{p-1} (-)^n t^n \Lambda^{\frac{1}{2}Kn(n-1)} c_n A_{n+1}(x) + S(t, m). \quad (6.33)$$

Here, the function $S(t, m)$ is defined by

$$\begin{aligned} S(t, m) &\equiv \sum_{n \neq 0, 1, \dots, p-1}^{\infty} (-)^n t^n \Lambda^{\frac{1}{2}Kn(n-1)} m^{\nu_n} c_n \\ \nu_n &\equiv N_1 + n(N_2 - N_1) - \frac{1}{2}Kn(n-1) \end{aligned} \quad (6.34)$$

and $S(t, m)$ is independent on the order parameters x_i^I .

The prepotential corresponding to (6.33) is again easily derived by taking limits in (5.23). We may define its basic building blocks $A_i^I(x_i^I)$, $B^I(x_i^I)$, $S_i^I(x_i^I)$ by exactly the same

expressions as in (6.17). Note however that for $1 < I < p$, the corresponding expression $B^I(x_i^I)$ results now in

$$B^I(x_i^I) = \prod_{j \in I-1} (\mu_{I-1} + x_j^{I-1} - x_i^I) \prod_{j \in I+1} (\mu_I + x_i^I - x_j^{I+1}).$$

If we introduce the constants ρ_I by

$$\rho_I = \prod_{\substack{j \in J \\ |J-I| \geq 2}} (1 - \frac{1}{(I-J)^2}),$$

we find the following limit for $i \in I$

$$qS_i(x_i^I) \rightarrow (-4)^{N_I} (\frac{\Lambda}{2})^K \rho_I S_i^I(x_i^I),$$

and hence the following expressions for the one- and two-instanton corrections

$$\begin{aligned} \mathcal{F}^{(1)} &= \frac{1}{2\pi i} (\frac{\Lambda}{2})^K \sum_I (-4)^{N_I} \rho_I \sum_{i \in I} S_i^I(x_i^I) \\ \mathcal{F}^{(2)} &= \frac{1}{2\pi i} (\frac{\Lambda}{2})^{2K} \sum_I 4^{2N_I} \rho_I^2 \left[\sum_{i \in I} S_i^I(x_i^I) (\frac{\partial}{\partial x_i^I})^2 S_i^I(x_i^I) + \sum_{\substack{i, j \in I \\ i \neq j}} \frac{S_i^I(x_i^I) S_j^I(x_j^I)}{(x_i^I - x_j^I)^2} \right]. \end{aligned}$$

Again, as in Section VI.(b), some of the parameters x_i^I may freeze to a constant expectation value, and the resulting $SU(N_I)$ should be viewed rather as a flavor group. We shall see below that this can take place only for $SU(N_p)$ and $SU(N_1)$.

The dynamics of the theory may be divided up into three categories, which are specified by the values of the parameters N_1 and N_p . Actually, the combinations that enter more naturally (and are equivalent to N_1 and N_p) are given by

$$K^+ \equiv 2N_p - N_{p-1} \quad K^- \equiv 2N_1 - N_2, \quad (6.35)$$

and (6.25) may equivalently be expressed in terms of these for $I = 2, \dots, p-1$

$$N_I = \frac{1}{2} K I (p+1-I) + \frac{1}{p+1} (I K^+ + (p+1-I) K^- - (p+1) K). \quad (6.36)$$

We now have the condition that $2(K^+ - K^-)$ be divisible by $p+1$ and that the resulting quotient be even (odd) if Kp is even (odd).

The function $S(t, m)$ of (6.34) will have a convergent limit if and only if ν_n of (6.34) satisfies

$$\nu_n \leq 0 \quad \text{for all } n \in \mathbf{Z}, \quad n \neq 0, 1, \dots, p-1. \quad (6.37)$$

Since ν_n is a downward parabola as a function of n , with $\nu_0 = N_1 > 0$, it is necessary and sufficient for (6.37) to hold that

$$\nu_{-1} = K^- - K \leq 0 \quad \nu_{+p} = K^+ - K \leq 0. \quad (6.38)$$

Assuming that $S(t, m)$ indeed converges to a finite limit, we are left with three inequivalent cases, which we now describe in turn.

(1) *The Case $K^+ = K^- = K$: Gauge Group $SU(N_1) \times \cdots \times SU(N_p)$ with Hypermultiplets in bi-Fundamental Representations*

The couplings of all $SU(N_I)$ have a finite limit as $m, \tau \rightarrow \infty$, and the full gauge group $SU(N_1) \times \cdots \times SU(N_p)$ remains. The formula for the N_I simplifies and we have $N_I = \frac{1}{2}KI(p+1-I)$ for all $I = 1, \dots, p$. The $p-1$ hypermultiplets that remain after decoupling are in bi-fundamental representations, given by

$$\sum_{I=1}^{p-1} \{(\mathbf{N}_I, \bar{\mathbf{N}}_{I+1}) \oplus (\bar{\mathbf{N}}_I, \mathbf{N}_{I+1})\}. \quad (6.39)$$

The limiting curve results from (6.33) with the finite limit of $S(t, m)$, which comes from the $n = -1$ and $n = p$ contributions only :

$$0 = \sum_{n=0}^{p-1} (-)^n t^n \Lambda^{\frac{1}{2}Kn(n-1)} c_n A_{n+1}(x) - \frac{1}{t} \Lambda^K c_{-1} + (-)^p t^p \Lambda^{\frac{1}{2}Kp(p-1)} c_p. \quad (6.40)$$

The cases $N_1 = N_2$, treated above are of this type with $p = 2$.

(2) *The Case $K^+ < K, K^- = K$: Gauge Group $SU(N_1) \times \cdots \times SU(N_{p-1})$ with Hypermultiplets in Fundamental & bi-Fundamental Representations*

The cases $K^+ < K, K^- = K$ and $K^- < K, K^+ = K$ are clearly equivalent, and we shall limit to dealing with the first. In that case, the limits of the couplings of $SU(N_1) \times \cdots \times SU(N_{p-1})$ are finite, while the coupling of $SU(N_p)$ converges to zero, so that the dynamics of this group gets frozen out. The group $SU(N_p)$ effectively becomes a flavor group and we have the following hypermultiplet contents : $p-2$ hypermultiplets in bi-fundamental representations, together with N_p hypermultiplets in the fundamental representations of each of the simple factors of the gauge group. In total, we have

$$\sum_{I=1}^{p-2} \{(\mathbf{N}_I, \bar{\mathbf{N}}_{I+1}) \oplus (\bar{\mathbf{N}}_I, \mathbf{N}_{I+1})\} \oplus \sum_{I=1}^{p-1} N_p \{\mathbf{N}_I \oplus \bar{\mathbf{N}}_I\} \quad (6.41)$$

The curve is easily obtained from (6.33) again and the limit now only contains a single term in the function $S(t, m)$, namely for $n = -1$. We find

$$0 = \sum_{n=0}^{p-1} (-)^n t^n \Lambda^{\frac{1}{2}Kn(n-1)} c_n A_{n+1}(x) - \frac{1}{t} \Lambda^K c_{-1} \quad (6.42)$$

The prepotential to 2 instanton order is again obtained by starting from the general formula (5.23) and taking the above limit $m \rightarrow \infty$ and $\tau \rightarrow \infty$.

(3) *The Case $K^-, K^+ < K$: Gauge Group $SU(N_2) \times \cdots \times SU(N_{p-1})$ with Hypermultiplets in Fundamental & bi-Fundamental Representations*

In this case, only the couplings of the gauge group $SU(N_2) \times \cdots \times SU(N_{p-1})$ converge to a finite value, while those of $SU(N_1) \times SU(N_p)$ converge to zero. Thus, $SU(N_1) \times SU(N_p)$ becomes an approximate flavor group. There are now $p-3$ hypermultiplets in bi-fundamental representations (for $p=2$, there are no such hypermultiplets), and $N_1 + N_p$ hypermultiplets in each of the fundamental representations of the simple factors of the gauge group. In total, we have

$$\sum_{I=2}^{p-2} \{(\mathbf{N}_I, \bar{\mathbf{N}}_{I+1}) \oplus (\bar{\mathbf{N}}_I, \mathbf{N}_{I+1})\} \oplus \sum_{I=2}^{p-1} (N_1 + N_p) \{\mathbf{N}_I \oplus \bar{\mathbf{N}}_I\} \quad (6.43)$$

The curve for this theory is

$$0 = \sum_{n=0}^{p-1} (-)^n t^n \Lambda^{\frac{1}{2}Kn(n-1)} c_n A_{n+1}(x) \quad (6.44)$$

(d) The Special Case $p=3$

It is very instructive to see how the general discussion of (c) can be applied to the special case $p=3$. For solutions to exist, it is necessary and sufficient that $K > 0$ and that the following general formal solutions be positive integers

$$\begin{aligned} N_1 &= \frac{1}{2}K + \frac{1}{4}(K^+ + 3K^-) \\ N_2 &= K + \frac{1}{2}(K^+ + K^-) \\ N_3 &= \frac{1}{2}K + \frac{1}{4}(3K^+ + K^-) \\ N &= 2K + \frac{3}{2}(K^+ + K^-). \end{aligned} \quad (6.45)$$

A particularly interesting limit is where $K^+, K^- < K$, so that the gauge groups $SU(N_1)$ and $SU(N_3)$ freeze out. Let us call the remaining gauge group color and set $N_c = N_2$ to be the number of colors. According to our general discussion, the freezing out of the gauge groups produces $N_f = N_1 + N_3 = K + K^+ + K^-$ flavors of fundamental representations of $SU(N_c)$. The quantity $2N_c - N_f = K$ had to be positive from general considerations and we now recognize this conditions as the criterion for asymptotic freedom ! Let's check that we obtain the curve (3.1). The curve equation that emerges from (6.44) is given by

$$c_0 A_1(x) - t c_1 A_2(x) + t^2 c_2 \Lambda^{2N_c - N_f} A_3(x) = 0 \quad (6.46)$$

This equation would appear to be different from (3.1). However, we shall now make the following change of variables

$$t = \frac{A_1(x)}{\tilde{y}} \frac{c_0}{c_1}$$

and define $A(x) = A_2(x)$, $B(x) = A_1(x)A_3(x)$, which puts the equation in a form

$$\tilde{y}^2 - \tilde{y}A(x) + \frac{c_0 c_2}{c_1^2} \Lambda^{2N_c - N_f} B(x) = 0. \quad (6.47)$$

This is equivalent to (3.1), upon setting $y = 2\tilde{y} - A(x)$ and redefining Λ by a multiplicative constant. The only restrictions on this result were $K^+, K^- < K$, so that the above construction is limited to $2N_c - N_f \geq 2$.

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